

# A DIFFERENTIAL OPERATOR AND TOM DIECK–KOSNIOWSKI–STONG LOCALIZATION THEOREM

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**ABSTRACT.** We define a differential operator on the “dual” algebra of the unoriented  $G$ -representation algebra introduced by Conner and Floyd, where  $G = (\mathbb{Z}_2)^n$ . With the help of  $G$ -colored graphs (or mod 2 GKM graphs), we may use this differential operator to give a very simply equivalent description of tom Dieck–Kosniowski–Stong localization theorem in the setting of smooth closed  $n$ -manifolds with effective smooth  $G$ -actions (also called  $n$ -dimensional 2-torus manifolds). We then apply this to study the  $G$ -equivariant unoriented cobordism classification of  $n$ -dimensional 2-torus manifolds. We show that the  $G$ -equivariant unoriented cobordism class of each  $n$ -dimensional 2-torus manifold contains an  $n$ -dimensional small cover as its representative, solving the conjecture posed in [15]. In addition, we also obtain that the graded noncommutative ring formed by the equivariant unoriented cobordism classes of all possible dimensional 2-torus manifolds is generated by the classes of small covers over the products of simplices.

## 1. INTRODUCTION

Throughout this paper, assume that  $G = (\mathbb{Z}_2)^n$  is the mod 2-torus group of rank  $n$ , and it naturally admits a linear space structure over  $\mathbb{Z}_2$ . Following [6]–[7], let  $\mathcal{Z}_m(G)$  denote the set of  $G$ -equivariant unoriented cobordism classes of all smooth closed  $m$ -dimensional manifolds with smooth  $G$ -actions fixing a finite set. Then  $\mathcal{Z}_*(G) = \sum_{m \geq 0} \mathcal{Z}_m(G)$  forms a graded commutative algebra over  $\mathbb{Z}_2$  with unit, where the addition and the multiplication are defined by the disjoint union and the cartesian product of  $G$ -manifolds respectively, and the unit in  $\mathcal{Z}_*(G)$  is given by the class of a single point with trivial  $G$ -action. Let  $\mathcal{R}_m(G)$  be the linear space over  $\mathbb{Z}_2$ , generated by the isomorphism classes of  $m$ -dimensional real  $G$ -representations. Then  $\mathcal{R}_*(G) = \sum_{m \geq 0} \mathcal{R}_m(G)$  becomes a graded commutative algebra over  $\mathbb{Z}_2$  with unit, called the unoriented  $G$ -representation algebra, where the multiplication is given by the direct sum of representations. It is well-known that each irreducible real  $G$ -representation is one-dimensional, so  $\mathcal{R}_*(G)$  is also the graded polynomial algebra over  $\mathbb{Z}_2$  generated by the isomorphism classes of all irreducible real  $G$ -representations. We know from [18] (see also [7] and [17]) that

$$\phi_* : \mathcal{Z}_*(G) \longrightarrow \mathcal{R}_*(G)$$

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defined by  $\{M\} \mapsto \sum_{p \in M^G} [\tau_p M]$  is a monomorphism as  $\mathbb{Z}_2$ -algebras where  $\tau_p M$  denotes the real  $G$ -representation on the tangent space at  $p \in M^G$ . Thus,  $\mathcal{Z}_*(G)$  is identified with a subalgebra  $\text{Im } \phi_*$  of  $\mathcal{R}_*(G)$ . Conner and Floyd showed in [6] (also see [7]) that when  $n = 1$   $\mathcal{Z}_*(G) \cong \mathbb{Z}_2$ , and when  $n = 2$ ,  $\mathcal{Z}_*(G) \cong \mathbb{Z}_2[u]$  where  $u$  denotes the class of  $\mathbb{R}P^2$  with the standard linear  $(\mathbb{Z}_2)^2$ -action. Implicitly we will have  $n > 2$  throughout.

In [9], tom Dieck showed that the equivariant unoriented cobordism class of any smooth closed  $G$ -manifold is completely determined by its equivariant Stiefel–Whitney characteristic numbers, and in particular, the existence of each  $G$ -manifold  $M^m$  in a class of  $\mathcal{Z}_m(G)$  can be characterized by the integral property of its fixed data (see [9, Theorem 6]). Later on, Kosniowski and Stong [13] gave a more precise formula for the characteristic numbers of  $M^m$  in terms of the fixed data. Combining their works and viewpoints gives the following localization theorem.

**Theorem 1.1** (tom Dieck–Kosniowski–Stong localization theorem). *Let  $\tau_1, \dots, \tau_l$  be  $l$  monomials in  $\mathcal{R}_m(G)$  such that each monomial contains no any isomorphism class of trivial irreducible real  $G$ -representation. Then a necessary and sufficient condition that  $\tau_1 + \dots + \tau_l \in \text{Im } \phi_m$  is that for any symmetric polynomial function  $f(x_1, \dots, x_m)$  over  $\mathbb{Z}_2$ ,*

$$(1.1) \quad \sum_{i=1}^l \frac{f(\chi^G(\tau_i))}{\chi^G(\tau_i)} \in H^*(BG; \mathbb{Z}_2)$$

where  $\chi^G(\tau_i)$  denotes the equivariant Euler class of  $\tau_i$ , which is a product of  $m$  nonzero elements of  $H^1(BG; \mathbb{Z}_2)$ , and  $f(\chi^G(\tau_i))$  means that variables  $x_1, \dots, x_m$  in  $f(x_1, \dots, x_m)$  are replaced by those  $m$  degree-one factors in  $\chi^G(\tau_i)$ .

*Remark 1.* It is well-known from [6]–[7] and [18] that each element of  $\text{Im } \phi_*$  is actually contained in the subalgebra of  $\mathcal{R}_*(G)$  generated by the isomorphism classes of nontrivial irreducible real  $G$ -representations. Thus, we see that all elements of  $\text{Im } \phi_*$  can be characterized by the formula (1.1). However, it is still quite difficult to determine the algebra structure of  $\text{Im } \phi_* \cong \mathcal{Z}_*(G)$  by using the formula (1.1).

Now let  $\text{Hom}(G, \mathbb{Z}_2)$  (resp.  $\text{Hom}(\mathbb{Z}_2, G)$ ) denote the set of all homomorphisms  $G \rightarrow \mathbb{Z}_2$  (resp.  $\mathbb{Z}_2 \rightarrow G$ ). Then both  $\text{Hom}(G, \mathbb{Z}_2)$  and  $\text{Hom}(\mathbb{Z}_2, G)$  have natural abelian group structures given by those of  $\mathbb{Z}_2$  and  $G$  in the usual way (i.e., the addition is given by  $(\rho + \xi)(g) = \rho(g) + \xi(g)$ ) and so they have also linear space structures over  $\mathbb{Z}_2$ . We can define an additional operation  $\circ$  on  $G$  by  $(g_1, \dots, g_n) \circ (g'_1, \dots, g'_n) = (g_1 g'_1, \dots, g_n g'_n)$ . Then it is easy to check that  $G$  has a commutative ring structure with respect to operations  $+$  and  $\circ$  (see also [5]). Let  $\mathbb{Z}_2[\text{Hom}(G, \mathbb{Z}_2)]$  (resp.  $\mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, G)]$ ) be the graded polynomial algebra over  $\mathbb{Z}_2$  generated by all elements of  $\text{Hom}(G, \mathbb{Z}_2)$  (resp.  $\text{Hom}(\mathbb{Z}_2, G)$ ) with the multiplication given by  $(\xi_1 \cdots \xi_k)(g) = (\underbrace{\xi_1(g) \cdots \xi_k(g), \dots, \xi_1(g) \cdots \xi_k(g)})$  (resp.

$$(\xi_1 \cdots \xi_k)(g) = (\underbrace{\xi_1(g) \circ \cdots \circ \xi_k(g), \dots, \xi_1(g) \circ \cdots \circ \xi_k(g)})^k \text{ where } k \geq 1 \text{ and } \xi_i \in$$

$\text{Hom}(G, \mathbb{Z}_2)$  (resp.  $\text{Hom}(\mathbb{Z}_2, G)$ ). Then we see that the structure on  $\mathbb{Z}_2[\text{Hom}(G, \mathbb{Z}_2)]$  (resp.  $\mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, G)]$ ) is compatible with the linear space structure on  $\text{Hom}(G, \mathbb{Z}_2)$  (resp.  $\text{Hom}(\mathbb{Z}_2, G)$ ). Therefore, we have that  $\mathbb{Z}_2[\text{Hom}(G, \mathbb{Z}_2)] \cong \mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, G)] \cong H^*(BG; \mathbb{Z}_2)$ .

On the other hand, it is well-known that all irreducible real  $G$ -representations correspond to all elements in  $\text{Hom}(G, \mathbb{Z}_2)$ , where every irreducible real representation of  $G$  has the form  $\lambda_\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$  with  $\lambda_\rho(g, x) = (-1)^{\rho(g)}x$  for  $\rho \in \text{Hom}(G, \mathbb{Z}_2)$ , and  $\lambda_\rho$  is trivial if  $\rho(g) = 0$  for all  $g \in G$ . Thus, if we forget any structure on  $\text{Hom}(G, \mathbb{Z}_2)$  and denote it by  $\widehat{\text{Hom}(G, \mathbb{Z}_2)}$ , then  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ , the graded polynomial algebra over  $\mathbb{Z}_2$  generated by all elements of  $\widehat{\text{Hom}(G, \mathbb{Z}_2)}$ , can be identified with  $\mathcal{R}_*(G)$ . Similarly, we may define  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  in the same way as  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . In a certain sense, both  $\widehat{\text{Hom}(G, \mathbb{Z}_2)}$  and  $\widehat{\text{Hom}(\mathbb{Z}_2, G)}$  are dual to each other. Thus, given a faithful  $G$ -polynomial  $g = \sum_i t_{i,1} \cdots t_{i,n}$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$  (which means that for each monomial  $t_{i,1} \cdots t_{i,n}$  of  $g$ , the set  $\{t_{i,1}, \dots, t_{i,n}\}$  is a basis of  $\text{Hom}(G, \mathbb{Z}_2)$ ), we can obtain a unique dual  $G$ -polynomial  $g^*$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  (see also Subsection 2.1). In Subsection 2.2 we shall define a differential operator  $d$  on  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ . Then the following result gives another characterization of  $g \in \text{Im } \phi_n$  in terms of  $d(g^*)$ .

**Theorem 1.2.** *Let  $g = \sum_i t_{i,1} \cdots t_{i,n}$  be a faithful  $G$ -polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . Then  $g \in \text{Im } \phi_n$  if and only if  $d(g^*) = 0$ .*

*Remark 2.* We shall use the  $G$ -colored graphs (or mod 2 GKM graphs) to give the proof of Theorem 1.2. See Subsection 3.1 for the  $G$ -colored graphs and related results.

Since each class  $\chi^G(\tau_i)$  uniquely corresponds to  $\tau_i$  in Theorem 1.1 and  $H^*(BG; \mathbb{Z}_2)$  is isomorphic to  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ , as a consequence of Theorems 1.1–1.2, we have

**Corollary 1.3.** *Let  $g = \sum_i t_{i,1} \cdots t_{i,n}$  be a faithful  $G$ -polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . Then  $d(g^*) = 0$  if and only if for any symmetric polynomial function  $f(x_1, \dots, x_n)$  over  $\mathbb{Z}_2$ ,*

$$\sum_i \frac{f(t_{i,1}, \dots, t_{i,n})}{t_{i,1} \cdots t_{i,n}} \in \mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}].$$

Our next task is to give an application of Theorem 1.2.

The first author of this paper in [15] introduced the abelian group  $\mathfrak{M}_n \subset \mathcal{Z}_*(G)$ , which consists of the  $G$ -equivariant unoriented cobordism classes of all  $n$ -dimensional smooth closed manifolds with effective smooth  $G$ -actions (note that  $\mathfrak{M}_n$  is also a linear space over  $\mathbb{Z}_2$ ). The structure of  $\mathfrak{M}_n$  was studied in [15], and it was shown therein that  $\dim_{\mathbb{Z}_2} \mathfrak{M}_1 = 0$ ,  $\dim_{\mathbb{Z}_2} \mathfrak{M}_2 = 1$  and  $\dim_{\mathbb{Z}_2} \mathfrak{M}_3 = 13$ . In addition, associated with the structure of  $\mathfrak{M}_n$ , the following conjecture was also posed in [15]:

**Conjecture (★):** *Each class of  $\mathfrak{M}_n$  contains a small cover as its representative.*

It has been shown in [15] that the Conjecture (★) holds for  $n \leq 3$ . However, the argument used in [15] does not work effectively in the case  $n > 3$ .

In this paper, Theorem 1.2 provides us a different approach to the study of  $\mathfrak{M}_n$ . Since the restriction of  $\phi_n$  to  $\mathfrak{M}_n$  is still a monomorphism, it follows by Theorem 1.2 that as linear spaces over  $\mathbb{Z}_2$ ,  $\mathfrak{M}_n$  is isomorphic to the linear space  $\mathcal{V}_n$  formed by all faithful  $G$ -polynomials  $g \in \mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$  with  $d(g^*) = 0$  (see Proposition 5.1). On the other hand, all  $\mathbb{Z}_2$ -homologies of the chain complex  $(\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}], d)$

vanish (see Proposition 2.1). Thus, the problem can be further reduced to studying the linear space  $\widehat{\mathcal{L}_n}$  formed by all squarefree homogeneous polynomials  $H$  of degree  $n+1$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  subject to the conditions: (1)  $d(H) \in \mathcal{V}_n^*$ ; (2) all  $n+1$  degree-one factors in each monomial of  $H$  contain a basis of  $\text{Hom}(\mathbb{Z}_2, G)$  (see Section 5). Based upon this, we will show that

**Theorem 1.4.** *The Conjecture  $(\star)$  holds for any dimension  $n$ .*

Let  $\mathfrak{M}_* = \sum_{n \geq 1} \mathfrak{M}_n$ . Then  $\mathfrak{M}_*$  forms a graded ring with the multiplication defined by  $\{M_1^{n_1}\} \cdot \{M_2^{n_2}\} = \{M_1^{n_1} \times M_2^{n_2}\}$  in such a natural way that the  $(\mathbb{Z}_2)^{n_1+n_2}$ -action on  $M_1^{n_1} \times M_2^{n_2}$  is given by  $((g_1, g_2), (x_1, x_2)) \mapsto (g_1(x_1), g_2(x_2))$  for  $x_i \in M_i^{n_i}$  and  $g_i \in (\mathbb{Z}_2)^{n_i}$  by regarding  $(\mathbb{Z}_2)^{n_1+n_2}$  as  $(\mathbb{Z}_2)^{n_1} \times (\mathbb{Z}_2)^{n_2}$ . It should be pointed out that the multiplication defined as above depends upon the ordering of the cartesian product of  $M_1^{n_1}$  with  $(\mathbb{Z}_2)^{n_1}$ -action and  $M_2^{n_2}$  with  $(\mathbb{Z}_2)^{n_2}$ -action. Actually, in the same way as above, by regarding  $(\mathbb{Z}_2)^{n_1+n_2}$  as  $(\mathbb{Z}_2)^{n_2} \times (\mathbb{Z}_2)^{n_1}$ , the  $(\mathbb{Z}_2)^{n_1+n_2}$ -action on  $M_2^{n_2} \times M_1^{n_1}$  would be defined by  $((g_2, g_1), (x_2, x_1)) \mapsto (g_2(x_2), g_1(x_1))$ . However, generally such two  $(\mathbb{Z}_2)^{n_1+n_2}$ -actions on  $M_1^{n_1} \times M_2^{n_2}$  and  $M_2^{n_2} \times M_1^{n_1}$  are not equivariantly cobordant except for  $\{M_1^{n_1}\} = \{M_2^{n_2}\}$ , but up to automorphisms of  $(\mathbb{Z}_2)^{n_1+n_2}$ , they have not any difference essentially (i.e., by using an automorphism of  $(\mathbb{Z}_2)^{n_1+n_2}$ , one of both can be changed into the other one). Thus,  $\mathfrak{M}_*$  is a graded noncommutative ring.

**Theorem 1.5.**  *$\mathfrak{M}_*$  is generated by the classes of all small covers over  $\Delta^{n_1} \times \cdots \times \Delta^{n_\ell}$  with  $n_1 + \cdots + n_\ell \geq 1$ , where  $\Delta^{n_i}$  is an  $n_i$ -simplex.*

We also determine the precise structure of  $\mathfrak{M}_4$ . In addition, we shall give a simple proof of the main result on  $\mathfrak{M}_3$  in [15] (see Remark 12).

**Proposition 1.6.**  *$\dim_{\mathbb{Z}_2} \mathfrak{M}_4 = 510$  and  $\mathfrak{M}_4$  is generated by merely the classes of small covers over  $\Delta^2 \times \Delta^2$ .*

This paper is organized as follows. In Section 2 we introduce the notions of faithful polynomials and its dual polynomials, and give the definition of the differential operator  $d$  on  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ . In Section 3 we review the basic theories of colored graphs and small covers. In particular, we also discuss the decomposability of  $G$ -colored simple convex polytopes. In Section 4 we give the proof of Theorem 1.2. In Section 5 we introduce the linear spaces  $\mathcal{V}_n$ ,  $\mathcal{V}_n^*$  and  $\mathcal{L}_n$ , and then use them to study the structure of  $\mathfrak{M}_n$  and to finish the proofs of Theorems 1.4–1.5. In Section 6 we give a summary on some essential relationships among 2-torus manifolds, coloring polynomials, colored simple convex polytopes, colored graphs, and also pose some problems.

Finally we conclude this section with the following remark on Theorem 1.1.

*Remark 3.* In Theorem 1.1, if  $\{\tau_1, \dots, \tau_l\}$  is the fixed data of a  $G$ -manifold  $M^m$ , then the polynomial (1.1) is exactly an equivariant Stiefel–Whitney number of  $M^m$ . Actually, if we formally write the equivariant total Stiefel–Whitney class of the tangent bundle  $\tau M^m$  as  $w^G(\tau M^m) = \prod_{i=1}^m (1 + x_i)$ , then the equivariant Stiefel–Whitney number  $f(x_1, \dots, x_m)[M^m]$  can be calculated by the formula

$$f(x_1, \dots, x_m)[M^m] = \sum_{i=1}^l \frac{f(\chi^G(\tau_i))}{\chi^G(\tau_i)} \in H^*(BG; \mathbb{Z}_2)$$

where  $[M^m]$  denotes the fundamental homology class of  $M^m$ . For more details, see [9] and [13].

## 2. FAITHFUL POLYNOMIALS, DUAL POLYNOMIALS AND A DIFFERENTIAL OPERATOR

**2.1. Faithful polynomials and dual polynomials.**  $\text{Hom}(G, \mathbb{Z}_2)$  and  $\text{Hom}(\mathbb{Z}_2, G)$  are clearly isomorphic to  $G$ , and they are dual to each other by the following pairing:

$$(2.1) \quad \langle \cdot, \cdot \rangle : \text{Hom}(\mathbb{Z}_2, G) \times \text{Hom}(G, \mathbb{Z}_2) \longrightarrow \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$$

defined by  $\langle \xi, \rho \rangle = \rho \circ \xi$ , composition of homomorphisms. For example, the standard basis  $\{\rho_1, \dots, \rho_n\}$  of  $\text{Hom}(G, \mathbb{Z}_2)$  gives the dual basis  $\{\rho_1^*, \dots, \rho_n^*\}$  of  $\text{Hom}(\mathbb{Z}_2, G)$ , where  $\rho_i$  is defined by  $(g_1, \dots, g_n) \mapsto g_i$ , and  $\rho_i^*$  is defined by  $a \mapsto \underbrace{(0, \dots, 0, a, 0, \dots, 0)}_{i-1}$ .

Suppose that  $g = \sum_i t_{i,1} \cdots t_{i,n}$  is a nonzero homogeneous polynomial of degree  $n$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$  such that each monomial  $t_{i,1} \cdots t_{i,n}$  is the class of an  $n$ -dimensional faithful  $G$ -representation so  $\{t_{i,1}, \dots, t_{i,n}\}$  forms a basis of  $\text{Hom}(G, \mathbb{Z}_2)$ . Then  $g$  is called a *faithful  $G$ -polynomial* of  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . By the pairing (2.1),  $\{t_{i,1}, \dots, t_{i,n}\}$  determines a dual basis  $\{s_{i,1}, \dots, s_{i,n}\}$  of  $\text{Hom}(\mathbb{Z}_2, G)$ . Furthermore, we obtain a unique homogeneous polynomial  $g^* = \sum_i s_{i,1} \cdots s_{i,n}$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ , which is called the *dual  $G$ -polynomial* of  $g$ .

*Example 2.1.* When  $n = 3$ , take a faithful polynomial  $g = \rho_1 \rho_2 \rho_3 + \rho_1 \rho_3 (\rho_2 + \rho_3) + \rho_1 \rho_2 (\rho_2 + \rho_3) + \rho_1 (\rho_1 + \rho_3) (\rho_1 + \rho_2) + \rho_1 (\rho_1 + \rho_3) (\rho_2 + \rho_3) + \rho_1 (\rho_1 + \rho_2) (\rho_2 + \rho_3)$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . Then the dual polynomial of  $g$  is  $g^* = \rho_1^* \rho_2^* \rho_3^* + \rho_1^* \rho_2^* (\rho_2^* + \rho_3^*) + \rho_1^* \rho_3^* (\rho_2^* + \rho_3^*) + \rho_2^* \rho_3^* (\rho_1^* + \rho_2^* + \rho_3^*) + \rho_2^* (\rho_2^* + \rho_3^*) (\rho_1^* + \rho_2^* + \rho_3^*) + \rho_3^* (\rho_2^* + \rho_3^*) (\rho_1^* + \rho_2^* + \rho_3^*)$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ .

**2.2. A differential operator  $d$  on  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ .** We define a differential operator  $d$  on  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  as follows: for each monomial  $s_1 \cdots s_i$  of degree  $i \geq 1$

$$d_i(s_1 \cdots s_i) = \begin{cases} \sum_{j=1}^i s_1 \cdots s_{j-1} \widehat{s_j} s_{j+1} \cdots s_i & \text{if } i > 1 \\ 1 & \text{if } i = 1. \end{cases}$$

and  $d_0(1) = 0$ , where the symbol  $\widehat{s_j}$  means that  $s_j$  is deleted. Obviously,  $d^2 = 0$ . Thus,  $(\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}], d)$  forms a chain complex.

**Proposition 2.1.** *For all  $i \geq 0$ ,  $H_i(\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]; \mathbb{Z}_2) = 0$ .*

*Proof.* It is easy to see that  $H_0(\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]; \mathbb{Z}_2) = 0$ . So it suffices to show that  $\text{Im} d_{i+1} = \ker d_i$  for  $i > 0$ . Obviously,  $\text{Im} d_{i+1} \subseteq \ker d_i$ . Conversely, for any  $h \in \ker d_i$ , take  $H = sh$  where  $s \in \widehat{\text{Hom}(\mathbb{Z}_2, G)}$ . Then  $d_{i+1}(H) = h + s d_i(h) = h$  so  $h \in \text{Im} d_{i+1}$ . Thus  $\text{Im} d_{i+1} \supseteq \ker d_i$ .  $\square$

**Definition 2.2.** A polynomial  $h \in \mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  is said to be *squarefree* if each monomial of  $h$  is a product of distinct nontrivial elements in  $\widehat{\text{Hom}(\mathbb{Z}_2, G)}$ , where the trivial element in  $\widehat{\text{Hom}(\mathbb{Z}_2, G)}$  is the zero homomorphism from  $\mathbb{Z}_2$  to  $G$ .

**Corollary 2.3.** *Let  $h \in \mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  be squarefree. Then  $d(h) = 0$  if and only if there is a squarefree polynomial  $H$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  such that  $d(H) = h$ .*

*Proof.* Obviously, if  $h = d(H)$ , then  $d(h) = d^2(H) = 0$ . Conversely, we see from the proof of Proposition 2.1 that for any nontrivial element  $s \in \widehat{\text{Hom}(\mathbb{Z}_2, G)}$ ,  $d(sh) = h + sd(h) = h$ . If  $sh$  is not squarefree, then we may write  $sh = sh_1 + s^2h_2$  such that  $sh_1$  is nonzero and squarefree. Furthermore,  $h = d(sh) = h_1 + sd(h_1) + s^2d(h_2)$ . This forces  $d(h_2)$  to be zero since  $h$  is squarefree. Thus, we can take  $H = sh_1$  as desired.  $\square$

*Remark 4.* It should be pointed out that similarly we may define a differential operator  $d'$  on  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . However, given a faithful polynomial  $g \in \mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ , if  $d(g^*) = 0$ , then generally we cannot obtain  $d'(g) = 0$ . Actually, a direct calculation shows that  $d(g^*) = 0$  but  $d'(g) \neq 0$  in Example 2.1.

Let  $h \in \mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ . For an automorphism  $\sigma$  of  $\text{Hom}(\mathbb{Z}_2, G)$ , let  $\sigma(h)$  denote the polynomial of  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  produced by replacing each degree-one factor  $t$  in  $h$  by  $\sigma(t)$ , where  $t$  is regarded as an element in  $\text{Hom}(\mathbb{Z}_2, G)$ . Then we see that  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  naturally admits an action  $\Phi$  of  $\text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ , defined by  $h \mapsto \sigma(h)$ . A direct calculation gives the following result.

**Lemma 2.4.** *Let  $h \in \mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  and  $\sigma \in \text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ . Then  $d(\sigma(h)) = \sigma(d(h))$ .*

### 3. $G$ -COLORED GRAPHS AND SMALL COVERS

**3.1.  $G$ -colored graphs.** In [10], Goresky, Kottwitz and MacPherson established the GKM theory, indicating that there is an essential link between topology and geometry of torus actions and the combinatorics of colored graphs (see also [12]). Such a link has already been expanded to the case of mod 2-torus actions (see, e.g., [1]–[2], [14], and [16]).

Following [14], let  $\Gamma$  be a finite regular graph of valence  $n$  without loops. If there is a map  $\alpha$  from the set  $E_\Gamma$  of all edges of  $\Gamma$  to all nontrivial elements of  $\text{Hom}(G, \mathbb{Z}_2)$  with the following properties:

- (1) for each vertex  $p$  of  $\Gamma$ ,  $\prod_{x \in E_p} \alpha(x)$  is faithful in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ , where  $E_p$  denotes the set of all edges adjacent to  $p$ ;
- (2) for each edge  $e$  of  $\Gamma$ ,  $\alpha(E_p) \equiv \alpha(E_q) \pmod{\alpha(e)}$  in  $\text{Hom}(G, \mathbb{Z}_2)$  where  $p$  and  $q$  are two endpoints of  $e$ ;

then the pair  $(\Gamma, \alpha)$  is called a  $G$ -colored graph of  $\Gamma$ , and  $\alpha$  is called a  $G$ -coloring on  $\Gamma$ .

*Remark 5.* It is easy to see that if all  $\alpha(E_p), p \in V_\Gamma$ , are distinct, then for each edge  $e \in E_\Gamma$ ,  $|E_e| = 1$  where  $V_\Gamma$  denotes the set of vertices in  $\Gamma$  and  $E_e$  denotes all edges joining two endpoints of  $e$  (see also [16, Lemma 5.1]).

Obviously, a  $G$ -colored graph  $(\Gamma, \alpha)$  gives a faithful  $G$ -polynomial

$$g_{(\Gamma, \alpha)} = \sum_{p \in V_\Gamma} \prod_{x \in E_p} \alpha(x)$$

in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ , which is also called the  $G$ -coloring polynomial of  $(\Gamma, \alpha)$ . It is known from [1] or [16, Section 2] that each  $G$ -manifold  $M$  in a class of  $\mathfrak{M}_n$  determines a  $G$ -colored graph  $(\Gamma_M, \alpha)$ , and the corresponding  $G$ -coloring polynomial

$g_{(\Gamma_M, \alpha)}$  is exactly  $\phi_n(\{M\})$ . On the other hand, we have known from [14, Proposition 2.2] that the  $G$ -coloring polynomial of each  $G$ -colored graph  $(\Gamma, \alpha)$  must belong to  $\text{Im } \phi_n$ . Therefore, it follows that

**Theorem 3.1.** *A faithful  $G$ -polynomial  $g \in \mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$  belongs to  $\text{Im } \phi_n$  if and only if it is the  $G$ -coloring polynomial of a  $G$ -colored graph  $(\Gamma, \alpha)$ .*

By  $\Lambda(G)$  we denote the set of all  $G$ -colored graphs  $(\Gamma, \alpha)$ .

**Definition 3.2.** Two  $G$ -colored graphs  $(\Gamma_1, \alpha_1)$  and  $(\Gamma_2, \alpha_2)$  in  $\Lambda(G)$  are said to be *equivalent* if  $g_{(\Gamma_1, \alpha_1)} = g_{(\Gamma_2, \alpha_2)}$ , denoted by  $(\Gamma_1, \alpha_1) \sim (\Gamma_2, \alpha_2)$ .

On the coset  $\Lambda(G)/\sim$ , define the addition  $+$  as follows:

$$\{(\Gamma_1, \alpha_1)\} + \{(\Gamma_2, \alpha_2)\} := \{(\Gamma_1, \alpha_1) \sqcup (\Gamma_2, \alpha_2)\}$$

where  $\sqcup$  means the disjoint union. Then  $\Lambda(G)/\sim$  forms an abelian group, where the zero element in  $\Lambda(G)/\sim$  is the class of the  $G$ -colored graph with zero  $G$ -coloring polynomial. By Theorem 3.1 we have that

**Proposition 3.3.**  $\mathfrak{M}_n$  is isomorphic to  $\Lambda(G)/\sim$ .

**Definition 3.4.** A  $G$ -colored graph  $(\Gamma, \alpha)$  in  $\Lambda(G)$  with  $g_{(\Gamma, \alpha)} \neq 0$  is said to be *prime* if all  $\alpha(E_p), p \in V_\Gamma$ , are distinct.

It is easy to see that a prime  $G$ -colored graph  $(\Gamma, \alpha)$  has the property that  $|V_\Gamma|$  equals to the number of monomials of  $g_{(\Gamma, \alpha)}$ . Now let us look at nonzero classes in  $\Lambda(G)/\sim$ .

**Lemma 3.5.** *Each nonzero class of  $\Lambda(G)/\sim$  contains a prime  $G$ -colored graph as its representative.*

*Proof.* Let  $(\Gamma, \alpha)$  be a  $G$ -colored graph in  $\Lambda(G)$  with  $g_{(\Gamma, \alpha)} \neq 0$ . Without the loss of generality assume that  $(\Gamma, \alpha)$  is not prime. Then there must be two vertices  $p$  and  $q$  such that  $\alpha(E_p) = \alpha(E_q)$ . Now let us perform a “connected sum” of  $(\Gamma, \alpha)$  to itself at  $p$  and  $q$  as follows: first cut out two vertices  $p$  and  $q$ , and then glue  $n$  edges  $\{e_p^1, \dots, e_p^n\}$  removed  $p$  to  $n$  edges  $\{e_q^1, \dots, e_q^n\}$  removed  $q$  along section endpoints respectively in such a way that two  $e_p^i$  and  $e_q^j$  will be glued together as long as  $\alpha(e_p^i) = \alpha(e_q^j)$ . Then it is easy to see that the resulting graph  $(\Gamma', \alpha')$  is still a  $G$ -colored graph in  $\Lambda(G)$ . This decreases two vertices  $p$  and  $q$  from  $(\Gamma, \alpha)$ , and clearly  $(\Gamma', \alpha') \sim (\Gamma, \alpha)$ . Since  $\Gamma$  is finite, this procedure can be ended until we obtain the desired prime  $G$ -colored graph.  $\square$

*Remark 6.* For two  $G$ -colored graphs  $(\Gamma_1, \alpha_1)$  and  $(\Gamma_2, \alpha_2)$ , if there are two vertices  $v_1 \in V_{\Gamma_1}$  and  $v_2 \in V_{\Gamma_2}$  such that  $\alpha_1(E_{v_1}) = \alpha_2(E_{v_2})$ , in a similar way as shown in the proof of Lemma 3.5, we can define a connected sum  $(\Gamma_1, \alpha_1) \#_{v_1, v_2} (\Gamma_2, \alpha_2)$  of  $(\Gamma_1, \alpha_1)$  and  $(\Gamma_2, \alpha_2)$  at  $v_1$  and  $v_2$ . Then we can obtain that

$$\{(\Gamma_1, \alpha_1) \#_{v_1, v_2} (\Gamma_2, \alpha_2)\} = \{(\Gamma_1, \alpha_1)\} + \{(\Gamma_2, \alpha_2)\} = \{(\Gamma_1, \alpha_1) \sqcup (\Gamma_2, \alpha_2)\}.$$

This also implies that for  $\{M_1\}, \{M_2\} \in \mathfrak{M}_n$ , if there are two fixed points  $p_1 \in M_1^G$  and  $p_2 \in M_2^G$  such that the tangent  $G$ -representations at  $p_1$  and  $p_2$  are isomorphic, then we can perform an equivariant connected sum  $M_1 \#_{p_1, p_2} M_2$  of  $M_1$  and  $M_2$  at  $p_1$  and  $p_2$ , and in particular,

$$\{M_1 \#_{p_1, p_2} M_2\} = \{M_1\} + \{M_2\} = \{M_1 \sqcup M_2\}.$$

**3.2. Small covers.** In [8] Davis and Januszkiewicz introduced and studied the topological version of real toric variety, i.e., “small cover”. This gives another link between the equivariant topology and the combinatorics of simple convex polytopes.

An  $n$ -dimensional small cover  $\pi : M^n \rightarrow P^n$  is a smooth closed  $n$ -manifold  $M^n$  with a locally standard  $G$ -action such that its orbit space is a simple convex  $n$ -polytope  $P^n$ . Each small cover  $\pi : M^n \rightarrow P^n$  determines a characteristic function  $\lambda$  (here we call it a  $G$ -coloring) on  $P^n$ , defined by mapping all facets (i.e.,  $(n-1)$ -dimensional faces) of  $P^n$  to nontrivial elements of  $\text{Hom}(\mathbb{Z}_2, G)$  such that  $n$  facets meeting at each vertex are mapped to  $n$  linearly independent elements. There is a fascinating characteristic of  $\pi : M^n \rightarrow P^n$ , saying that the algebraic topology of  $M^n$  is essentially consistent with the algebraic combinatorics of  $(P^n, \lambda)$  in many aspects. For example, the mod 2 Betti numbers  $(b_0, b_1, \dots, b_n)$  of  $M^n$  agree with the  $h$ -vector  $(h_0, h_1, \dots, h_n)$  of  $P^n$ . This leads us to one of reasons why we pose the Conjecture  $(\star)$ . The other one is that in [4] Buchstaber and Ray gave the proof of the Conjecture  $(\star)$  in non-equivariant case, i.e., each  $n$ -dimensional class of  $\mathfrak{N}_*$  contains a small cover as its representative, where  $\mathfrak{N}_*$  denotes the Thom unoriented coborism ring.

Now suppose that  $\pi : M^n \rightarrow P^n$  is a small cover, and  $\lambda : \mathcal{F}(P^n) \rightarrow \text{Hom}(\mathbb{Z}_2, G)$  is its characteristic function, where  $\mathcal{F}(P^n)$  consists of all facets of  $P^n$ . By the definition of  $\lambda$ , we see easily that each vertex  $v$  of  $P^n$  determines a monomial (called the  $G$ -coloring monomial of  $v$  and denoted by  $\lambda_v$ ) of degree  $n$  in  $\mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, G)]$ , whose dual by the pairing (2.1) is faithful in  $\mathbb{Z}_2[\text{Hom}(G, \mathbb{Z}_2)]$ . Moreover, all vertices  $V_{P^n}$  of  $P^n$  via  $\lambda$  give a polynomial  $\sum_{v \in V_{P^n}} \lambda_v$  of degree  $n$  in  $\mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, G)]$ , denoted by  $g_{(P^n, \lambda)}$ . Here we call  $g_{(P^n, \lambda)}$  the  $G$ -coloring polynomial of  $(P^n, \lambda)$ . Geometrically, each degree-one factor  $\delta$  of the monomial  $\lambda_v$  at vertex  $v$  is actually the normal representation to the characteristic submanifold fixed by the  $\mathbb{Z}_2$ -subgroup of  $G$  corresponding to the factor  $\delta$ . Since  $v$  is in the intersection of those  $n$  characteristic submanifolds determined by  $n$  degree-one factors of  $\lambda_v$ , this means that the dual of  $\lambda_v$  by the pairing (2.1) is exactly the tangent  $G$ -representation at the fixed point  $\pi^{-1}(v)$  (see [14, Proposition 4.1]). This is also shown in [3] for the case of quasi-toric manifolds in terms of matrices. Now let  $(\Gamma_M, \alpha)$  be the  $G$ -colored graph of  $\pi : M^n \rightarrow P^n$ , and let  $g_{(\Gamma_M, \alpha)}$  be the  $G$ -coloring polynomial of  $(\Gamma_M, \alpha)$ . Then we have

**Proposition 3.6.**  $g_{(P^n, \lambda)}$  is the dual polynomial of  $g_{(\Gamma_M, \alpha)}$ .

*Remark 7.* We know from [14, Proposition 4.1; Remark 4] that  $\Gamma_M$  is exactly the 1-skeleton of  $P^n$ , and both  $\lambda$  and  $\alpha$  are dual to each other. Given an automorphism  $\sigma \in \mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, G)]$ , from  $\lambda : \mathcal{F}(P^n) \rightarrow \text{Hom}(\mathbb{Z}_2, G)$  we can induce a new  $G$ -coloring  $\sigma \circ \lambda$  on  $P^n$ . Furthermore, we may see easily that  $g_{(P^n, \sigma \circ \lambda)} = \sigma(g_{(P^n, \lambda)})$ .

The following is a generalization of  $G$ -colorings on simple convex  $n$ -polytopes.

**Definition 3.7.** Let  $P^k$  be a simple convex  $k$ -polytope with  $k \leq n$ . A  $G$ -coloring  $\lambda$  on  $P^k$  is a map from all facets of  $P^k$  to  $\text{Hom}(\mathbb{Z}_2, G)$  such that  $\lambda$  maps  $k$  facets at each vertex of  $P^k$  into  $k$  linearly independent elements in  $\text{Hom}(\mathbb{Z}_2, G)$ . Let  $\ell \leq n - k$ .  $\lambda$  modulo  $(\mathbb{Z}_2)^\ell$  means that each facet  $F$  will be mapped into an element  $a \in \text{Hom}(\mathbb{Z}_2, G) / \text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^\ell) \cong \text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^{n-\ell})$  such that  $a \equiv \lambda(F)$  modulo  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^\ell)$ .



Since  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k) \subseteq \text{Hom}(\mathbb{Z}_2, G)$  for  $k \leq n$ , each  $(\mathbb{Z}_2)^k$ -coloring on  $P^k$  can always be regarded as a  $G$ -coloring on  $P^k$ .

Let  $P_i^{n_i}$  be a simple convex polytope of dimension  $n_i$ ,  $i = 1, 2$ , and let  $n_1 + n_2 = n$ . Regarded  $(\mathbb{Z}_2)^{n_1}$  as a subgroup of  $G = (\mathbb{Z}_2)^{n_1+n_2}$ , if  $\lambda_1$  is a  $(\mathbb{Z}_2)^{n_1}$ -coloring on  $P_1^{n_1}$  but  $\lambda_2$  is a  $G$ -coloring on  $P_2^{n_2}$  such that  $\lambda_2$  modulo  $(\mathbb{Z}_2)^{n_1}$  is a  $(\mathbb{Z}_2)^{n_2}$ -coloring on  $P_2^{n_2}$ , then we can define a  $G$ -coloring  $\lambda$  on  $P_1^{n_1} \times P_2^{n_2}$  as follows: for each facet  $F_i$  of  $P_i^{n_i}$ ,

$$\lambda(F) = \begin{cases} \lambda_1(F_1) & \text{if } F = F_1 \times P_2^{n_2} \\ \lambda_2(F_2) & \text{if } F = P_1^{n_1} \times F_2. \end{cases}$$

Note that all facets of  $P_1^{n_1} \times P_2^{n_2}$  consist of the polytopes of the forms  $F_1 \times P_2^{n_2}$  and  $P_1^{n_1} \times F_2$ . An easy observation shows that

**Lemma 3.8** (Product formula).

$$g_{(P_1^{n_1} \times P_2^{n_2}, \lambda)} = g_{(P_1^{n_1}, \lambda_1)} g_{(P_2^{n_2}, \lambda_2)}.$$

On the other hand, suppose that a simple convex  $n$ -polytope  $P^n$  admits a  $G$ -coloring  $\lambda$ . If  $P^n$  is decomposable (i.e.,  $P^n$  is the product  $P_1 \times P_2$  of two simple convex polytopes  $P_1$  and  $P_2$ ), then each  $P_i$  naturally inherits a  $G$ -coloring  $\lambda_i$  in such a way that for each facet  $F$  of  $P_i$ ,

$$\lambda_i(F) = \begin{cases} \lambda(F \times P_2) & \text{if } F \text{ is a facet of } P_1 \\ \lambda(P_1 \times F) & \text{if } F \text{ is a facet of } P_2. \end{cases}$$

Furthermore, it is easy to see that the product formula in Lemma 3.8 still holds in this case. Namely,  $g_{(P^n, \lambda)} = g_{(P_1, \lambda_1)} g_{(P_2, \lambda_2)}$ . This also implies the following result.

**Corollary 3.9.** *Suppose that a simple convex  $n$ -polytope  $P^n$  admits a  $G$ -coloring  $\lambda$  such that  $g_{(P^n, \lambda)}$  is indecomposable in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ . Then  $P^n$  is indecomposable too.*

Throughout the following we use the convention that all simple convex  $n$ -polytopes are embedded in  $\mathbb{R}^n$ , and if two simple convex polytopes  $P_1^n$  and  $P_2^n$  are combinatorially equivalent, then  $P_1^n$  is identified with  $P_2^n$ .

Now suppose that  $(P_1^n, \lambda_1)$  and  $(P_2^n, \lambda_2)$  are two  $G$ -colored simple convex  $n$ -polytopes such that there are two vertices  $v_1 \in P_1^n$  and  $v_2 \in P_2^n$  with  $\lambda_{1v_1} = \lambda_{2v_2}$ . Then we can always perform a connected sum  $P_1^n \#_{v_1, v_2} P_2^n$  of  $(P_1^n, \lambda_1)$  and  $(P_2^n, \lambda_2)$  at  $v_1$  and  $v_2$ , so that  $P_1^n \#_{v_1, v_2} P_2^n$  is still a simple convex polytope. In fact, because  $(P_i^n, \lambda_i)$  is identified with its mirror reflection  $(\overline{P}_i^n, \overline{\lambda}_i)$  along a hyperplane in  $\mathbb{R}^n$  but the  $G$ -coloring order of  $n$  facets at  $v_i$  in  $(P_i^n, \lambda_i)$  is exactly the reversion of the  $G$ -coloring order of  $n$  facets at  $\overline{v}_i$  in  $(\overline{P}_i^n, \overline{\lambda}_i)$  where  $\overline{v}_i$  is the reflection point of  $v_i$ , this means that we can choose  $(P_1^n, \lambda_1)$  and  $(P_2^n, \lambda_2)$  up to combinatorial equivalence (if necessary) such that the  $G$ -coloring order of  $n$  facets at  $v_1$  in  $(P_1^n, \lambda_1)$  is the reversion of the  $G$ -coloring order of  $n$  facets at  $v_2$  in  $(P_2^n, \lambda_2)$ . Thus, we can perform the required connected sum between  $(P_1^n, \lambda_1)$  and  $(P_2^n, \lambda_2)$ . Next it is not difficult to see that  $\lambda_1$  and  $\lambda_2$  determine a  $G$ -coloring  $\lambda$  on  $P_1^n \#_{v_1, v_2} P_2^n$ . Moreover, we have that

**Lemma 3.10** (Connected sum formula).

$$g_{(P_1^n \#_{v_1, v_2} P_2^n, \lambda)} = g_{(P_1^n, \lambda_1)} + g_{(P_2^n, \lambda_2)}.$$

*Proof.* This is because  $g_{(P_1^n \#_{v_1, v_2} P_2^n, \lambda)} = (g_{(P_1^n, \lambda_1)} - \lambda_{1v_1}) + (g_{(P_2^n, \lambda_2)} - \lambda_{2v_2}) = g_{(P_1^n, \lambda_1)} + g_{(P_2^n, \lambda_2)}$  in  $\mathbb{Z}_2[\widehat{\text{Hom}}(\mathbb{Z}_2, G)]$ .  $\square$

#### 4. PROOF OF THEOREM 1.2

First let us state a useful lemma.

**Lemma 4.1.** *If  $t\eta_2 \cdots \eta_n$  and  $t\theta_2 \cdots \theta_n$  are two different faithful  $G$ -monomials of degree  $n$  in  $\mathbb{Z}_2[\widehat{\text{Hom}}(G, \mathbb{Z}_2)]$  with  $\{\eta_2, \dots, \eta_n\} \equiv \{\theta_2, \dots, \theta_n\} \pmod{t}$  in  $\text{Hom}(G, \mathbb{Z}_2)$ , then their duals are different and contain the same monomial of degree  $n-1$  as a factor. Conversely, this is also true.*

*Proof.* This follows immediately from the pairing (2.1).  $\square$

**Proposition 4.2.** *Let  $g$  be a faithful  $G$ -polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}}(G, \mathbb{Z}_2)]$ . If  $g \in \text{Im } \phi_n$  then  $d(g^*) = 0$ .*

*Proof.* Let  $g \in \text{Im } \phi_n$ . Then by Theorem 3.1, Proposition 3.3 and Lemma 3.5, there is a prime  $G$ -colored graph  $(\Gamma, \alpha)$  with  $g_{(\Gamma, \alpha)} = g$ . Take an edge  $e = pq$  in  $\Gamma$ , we have that  $\alpha(E_p) \setminus \{\alpha(e)\} \equiv \alpha(E_q) \setminus \{\alpha(e)\} \pmod{\alpha(e)}$  in  $\text{Hom}(G, \mathbb{Z}_2)$ . Moreover, by Lemma 4.1, it follows that all monomials of degree  $n-1$  in  $d(g^*)$  appear in pairs, so  $d(g^*) = 0$ .  $\square$

**Proposition 4.3.** *Suppose that  $g = \sum_{i=1}^{\ell} t_{i,1} \cdots t_{i,n}$  is a faithful  $G$ -polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}}(G, \mathbb{Z}_2)]$ . If  $d(g^*) = 0$ , then  $g$  is the  $G$ -coloring polynomial of a  $G$ -colored graph.*

*Proof.* Assume  $d(g^*) = 0$ . We directly construct the required  $G$ -colored graph as follows:

First, write  $g^* = \sum_{i=1}^{\ell} s_{i,1} \cdots s_{i,n}$  such that each  $s_{i,1} \cdots s_{i,n}$  is the dual monomial of  $t_{i,1} \cdots t_{i,n}$ , and then take  $\ell$  points  $v_1, \dots, v_{\ell}$  as vertices, which are labeled by monomials  $s_{1,1} \cdots s_{1,n}, \dots, s_{\ell,1} \cdots s_{\ell,n}$ , respectively.

Next, for each vertex  $v_i$ , the monomial  $s_{i,1} \cdots s_{i,n}$  corresponding to  $v_i$  contains  $n$  monomials  $s_{i,1} \cdots \widehat{s}_{i,j} \cdots s_{i,n}$  of degree  $n-1$ ,  $j = 1, \dots, n$ . These  $n$  monomials of degree  $n-1$  are distinct since  $g$  is faithful. Then we can use  $n$  segments to make a bouquet with  $v_i$  as a common endpoint by gluing only an endpoint of each segment to  $v_i$ , and further we use the  $n$  monomials  $s_{i,1} \cdots \widehat{s}_{i,j} \cdots s_{i,n}$  of degree  $n-1$  ( $j = 1, \dots, n$ ) to label these  $n$  segments. So we can exactly get  $\ell$  such bouquets with  $v_1, \dots, v_{\ell}$  as their common endpoints respectively. Since  $d(g^*) = \sum_{i=1}^{\ell} \sum_{j=1}^n s_{i,1} \cdots \widehat{s}_{i,j} \cdots s_{i,n} = 0$ , this means that all  $s_{i,1} \cdots \widehat{s}_{i,j} \cdots s_{i,n}$  ( $i = 1, \dots, \ell, j = 1, \dots, n$ ) exactly appear in pairs. Moreover, we may use those  $\ell$  labeled bouquets to produce an  $n$ -valent regular graph  $\Gamma$  with  $\{v_1, \dots, v_{\ell}\}$  as its vertex set by pairwise gluing segments with same labels together along non-common endpoints of bouquets.

Our next procedure is to do a change of labels on all edges of  $\Gamma$ . We see easily from the pairing (2.1) that each factor  $t_{i,j}$  in  $t_{i,1} \cdots t_{i,n}$  uniquely corresponds to a monomial of degree  $n-1$  in  $s_{i,1} \cdots s_{i,n}$ . Without the loss of generality, we may assume that  $t_{i,j}$  exactly corresponds to  $s_{i,1} \cdots \widehat{s}_{i,j} \cdots s_{i,n}$ . Then we can give new labels on all edges of  $\Gamma$  by using  $t_{i,j}$  to replace  $s_{i,1} \cdots \widehat{s}_{i,j} \cdots s_{i,n}$  ( $i = 1, \dots, \ell, j = 1, \dots, n$ ). Moreover, we conclude by Lemma 4.1 that the graph  $\Gamma$  with new labels on edges is exactly the required  $G$ -colored graph.  $\square$

Together with Theorem 3.1 and Propositions 4.2–4.3, we complete the proof of Theorem 1.2.

### 5. THE CONJECTURE $(\star)$ AND THE BASIC STRUCTURE OF $\mathfrak{M}_\star$

Now let  $\mathcal{V}_n$  be the linear space over  $\mathbb{Z}_2$  produced by all faithful  $G$ -polynomials  $g$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$  with  $d(g^*) = 0$ . Then we have from Theorems 1.2, 3.1 and Proposition 3.3 that

**Proposition 5.1.**  *$\mathfrak{M}_n$  is isomorphic to  $\mathcal{V}_n$ .*

*Remark 8.* By  $\mathcal{V}_n^*$  we denote the linear space over  $\mathbb{Z}_2$  formed by the dual polynomials of those polynomials in  $\mathcal{V}_n$ . Then  $\mathcal{V}_n^*$  is clearly isomorphic to  $\mathcal{V}_n$ .

Let  $g$  be a faithful  $G$ -polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . By Corollary 2.3, we have that  $d(g^*) = 0$  if and only if there is a squarefree homogeneous polynomial  $H$  of degree  $n+1$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  such that  $d(H) = g^*$  and all  $n+1$  degree-one factors in each monomial of  $H$  contain a basis of  $\text{Hom}(\mathbb{Z}_2, G)$ . By  $\mathcal{L}_n$  we denote the linear space over  $\mathbb{Z}_2$  formed by all squarefree homogeneous polynomials  $H$  of degree  $n+1$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  satisfying the conditions:

- (a) for each such  $H$ ,  $d(H) \in \mathcal{V}_n^*$ ;
- (b) all  $n+1$  degree-one factors in each monomial of  $H$  contain a basis of  $\text{Hom}(\mathbb{Z}_2, G)$ .

Then we have that

**Lemma 5.2.**  *$\Psi : \mathcal{L}_n \longrightarrow \mathcal{V}_n^*$  defined by  $\Psi(H) = d(H)$  for  $H \in \mathcal{L}_n$  is an epimorphism.*

*Remark 9.* Proposition 5.1 and Lemma 5.2 provide a way to study the structure of  $\mathfrak{M}_n$  via  $\mathcal{V}_n(\cong \mathcal{V}_n^*)$ ,  $\mathcal{L}_n$  and  $\Psi$ .

Let  $H \in \mathcal{L}_n$ . For an automorphism  $\sigma$  of  $\text{Hom}(\mathbb{Z}_2, G)$ , it is easy to see that  $\sigma(H) \in \mathcal{L}_n$ . Thus, as a subspace of  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ ,  $\mathcal{L}_n$  is stable (or invariant) under the action  $\Phi$  of  $\text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$  (for the definition of  $\Phi$ , see Subsection 2.2). By Lemma 2.4, we have that  $d(\sigma(H)) = \sigma(d(H))$  for  $H \in \mathcal{L}_n$ . Thus, by Lemma 5.2,  $\mathcal{V}_n^*$  is also stable under the action  $\Phi$  of  $\text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ . This gives

**Lemma 5.3.**  *$\mathcal{L}_n$  and  $\mathcal{V}_n^*$  are stable under the action  $\Phi$  of  $\text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ .*

**5.1. Structure of  $\mathcal{L}_n$ .** Recall that  $\{\rho_1^*, \dots, \rho_n^*\}$  is a basis of  $\text{Hom}(\mathbb{Z}_2, G)$  where each  $\rho_i^*$  is defined by  $a \mapsto (0, \dots, 0, a, 0, \dots, 0)$  (see Subsection 2.1). Given a polynomial

$H$  in  $\mathcal{L}_n$ , let  $s_1 \cdots s_{n+1}$  be a monomial of  $H$ . Since  $s_1, \dots, s_{n+1}$  contain a basis of  $\text{Hom}(\mathbb{Z}_2, G)$ , we can apply an automorphism of  $\text{Hom}(\mathbb{Z}_2, G)$  to change this basis into  $\{\rho_1^*, \dots, \rho_n^*\}$ , so without the loss of generality we may assume that  $s_1, \dots, s_{n+1}$  exactly contain the basis  $\{\rho_1^*, \dots, \rho_n^*\}$  of  $\text{Hom}(\mathbb{Z}_2, G)$ . Furthermore, it is easy to see that up to the automorphisms of  $\text{Hom}(\mathbb{Z}_2, G)$ , all possible choices of  $s_1 \cdots s_{n+1}$  are the following

$$\begin{cases} \rho_1^* \cdots \rho_n^* (\rho_1^* + \cdots + \rho_n^*) \\ \rho_1^* \cdots \rho_n^* (\rho_1^* + \cdots + \rho_{n-1}^*) \\ \vdots \\ \rho_1^* \cdots \rho_n^* (\rho_1^* + \rho_2^*). \end{cases}$$

Obviously,  $\rho_1^* \cdots \rho_n^*(\rho_1^* + \cdots + \rho_n^*)$  belongs to  $\mathcal{L}_n$ . If  $H$  contains the monomial  $\rho_1^* \cdots \rho_n^*(\rho_1^* + \cdots + \rho_{n-1}^*)$ , then it must contain the monomial  $\rho_1^* \cdots \rho_{n-1}^*(\rho_1^* + \cdots + \rho_{n-1}^*)(\rho_n^* + s)$  where  $s \neq 0$  is a linear combination of  $\rho_1^*, \dots, \rho_{n-1}^*$ . Then we see that  $\rho_1^* \cdots \rho_n^*(\rho_1^* + \cdots + \rho_{n-1}^*) + \rho_1^* \cdots \rho_{n-1}^*(\rho_1^* + \cdots + \rho_{n-1}^*)(\rho_n^* + s)$  is in  $\mathcal{L}_n$ .

Generally, if  $H$  contains the monomial  $\rho_1^* \cdots \rho_n^*(\rho_1^* + \cdots + \rho_i^*)$  with  $i > 1$ , then we can write

$$H = \rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)h + H'$$

where  $h$  is a homogeneous polynomial of degree  $n-i$  such that all degree-one factors of each monomial of  $h$  are linearly independent, and each monomial of  $H'$  contains no  $\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)$ . Since

$$d(H) = \rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)d(h) + hd(\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)) + d(H')$$

belongs to  $\mathcal{V}_n^*$ , this forces  $d(h)$  to be zero. Thus, we conclude that

$$\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)h \in \mathcal{L}_n.$$

In particular, by reducing modulo  $\rho_1^*, \dots, \rho_i^*$ , all degree-one factors of each monomial of  $h$  are also linearly independent in  $\text{Hom}(\mathbb{Z}_2, G)/\text{Span}\{\rho_1^*, \dots, \rho_i^*\}$  because  $d(\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)h) = hd(\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)) \in \mathcal{V}_n^*$ .

Combining the above argument gives the following result.

**Lemma 5.4.**  $\mathcal{L}_n$  is generated by the polynomials of the following form

$$\sigma(H_i), \quad 1 < i \leq n$$

where  $\sigma \in \text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ , and  $H_i = \rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)h_i$  with  $h_i$  having the property that  $h_i$  is a nonzero homogeneous polynomial of degree  $n-i$  such that  $d(h_i) = 0$ , and if  $i < n$ , by reducing modulo  $\rho_1^*, \dots, \rho_i^*$ , all degree-one factors of each monomial of  $h_i$  are linearly independent in  $\text{Hom}(\mathbb{Z}_2, G)/\text{Span}\{\rho_1^*, \dots, \rho_i^*\}$ .

*Remark 10.* It is easy to see that for  $1 < i \leq n$ ,  $\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)$  may correspond to the colored  $i$ -dimensional simplex  $(\Delta^i, \lambda_0)$  such that  $i+1$  facets of  $\Delta^i$  can be labeled by  $\rho_1^*, \dots, \rho_i^*, (\rho_1^* + \cdots + \rho_i^*)$ , respectively. Then the coloring polynomial of  $(\Delta^i, \lambda_0)$  is  $d(\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*))$ .

**5.2. Structure of  $\mathcal{V}_n^*$  and proofs of Theorems 1.4–1.5.** The information from Lemma 5.4 and Remark 10 can provide us much insight into the further study of  $\mathcal{V}_n^*$ .

**Definition 5.5.** Let  $f$  be a squarefree homogeneous polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  with  $0 < \deg f \leq n$ . We say that  $f$  is  $G$ -colorable if  $f$  is the sum of the coloring polynomials of some  $G$ -colored simple convex  $\deg f$ -dimensional polytopes  $(P_1, \lambda_1), \dots, (P_l, \lambda_l)$ , where each  $P_i$  is a product of simplices.

*Remark 11.* By definition 5.5, it is easy to see that the sum of any two  $G$ -colorable polynomials is also  $G$ -colorable. Furthermore, if  $f$  is  $G$ -colorable, it follows from Remark 7 that for any automorphism  $\sigma \in \text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ ,  $\sigma(f)$  is  $G$ -colorable.

Let  $\mathcal{S}$  denote the set formed by those squarefree homogeneous polynomials  $h \in \mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  with the following properties:

- (1)  $d(h) = 0$ ;
- (2)  $0 < \deg h \leq n-2$ ;

- (3) each  $h$  is associated with a subspace  $V_h$  of dimension  $n - \deg h$  in  $\text{Hom}(\mathbb{Z}_2, G)$  such that by reducing modulo a basis of  $V_h$ , all degree-one factors of each monomial of  $h$  are linearly independent in  $\text{Hom}(\mathbb{Z}_2, G)/V_h$ .

**Lemma 5.6.** *Let  $h$  be a polynomial in  $\mathcal{S}$ . Then  $h$  is  $G$ -colorable.*

*Proof.* We shall perform an induction on  $\deg h$ . If  $\deg h = 1$ , since  $d(h) = 0$ , then there must be an even number of distinct monomials  $s_1, \dots, s_{2r}$  of degree one in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  such that  $h = s_1 + \dots + s_{2r}$ . Each pair  $(s_{2i-1}, s_{2i})$  can give a  $G$ -coloring  $\lambda_i$  on  $\Delta^1$  with two facets colored by  $s_{2i-1}, s_{2i}$  respectively, so that the coloring polynomial  $g_{(\Delta^1, \lambda_i)}$  of  $(\Delta^1, \lambda_i)$  is  $s_{2i-1} + s_{2i}$ , where  $1 \leq i \leq r$ . Thus  $h = g_{(\Delta^1, \lambda_1)} + \dots + g_{(\Delta^1, \lambda_r)}$ , and so  $h$  is  $G$ -colorable.

Now suppose inductively that  $h$  is  $G$ -colorable if  $\deg h \leq k$ . Consider the case in which  $\deg h = k + 1$ . Without the loss of generality, assume that  $V_h$  is generated by  $\rho_1^*, \dots, \rho_{n-k-1}^*$ , and some monomials of  $h$  contain degree-one factors of the form  $s$  with  $s \equiv \rho_{n-k}^* \pmod{V_h}$ . Then we may write  $h$  as

$$(5.1) \quad h = s_1 a_1 + \dots + s_r a_r + h'$$

where  $s_1, \dots, s_r$  are distinct degree-one elements in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$  such that  $s_i \equiv \rho_{n-k}^* \pmod{V_h}$ ,  $i = 1, \dots, r$ , and after reducing modulo  $V_h$ ,  $h'$  contains no degree-one factor  $\rho_{n-k}^*$ . Since  $d(h) = 0$ , by a direct calculation we have that

$$(s_1 d(a_1) + \dots + s_r d(a_r)) + (a_1 + \dots + a_r + d(h')) = 0.$$

Since  $a_1 + \dots + a_r + d(h')$  contains no degree-one factors  $s_1, \dots, s_r$ , we can conclude that

$$s_1 d(a_1) + \dots + s_r d(a_r) = a_1 + \dots + a_r + d(h') = 0$$

in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ . From  $s_1 d(a_1) + \dots + s_r d(a_r) = 0$ , an easy argument shows that  $d(a_i) = 0$ ,  $i = 1, \dots, r$  and further  $a_i \in \mathcal{S}$ ,  $i = 1, \dots, r$ . From  $a_1 + \dots + a_r + d(h') = 0$ , we have that  $a_r = a_1 + \dots + a_{r-1} + d(h')$ , so (5.1) becomes

$$\begin{aligned} h &= s_1 a_1 + \dots + s_{r-1} a_{r-1} + s_r (a_1 + \dots + a_{r-1} + d(h')) + h' \\ &= (s_1 + s_r) a_1 + \dots + (s_{r-1} + s_r) a_{r-1} + s_r d(h') + h' \\ &= (s_1 + s_r) a_1 + \dots + (s_{r-1} + s_r) a_{r-1} + d(s_r h'). \end{aligned}$$

Since  $\deg a_i = k < k + 1$  and  $a_i \in \mathcal{S}$ , by induction hypothesis, we have that each  $a_i$  is  $G$ -colorable. Obviously,  $s_1 + s_r, \dots, s_{r-1} + s_r$  are all  $G$ -colorable. Thus, by Lemma 3.8, it follows that  $(s_1 + s_r) a_1 + \dots + (s_{r-1} + s_r) a_{r-1}$  is  $G$ -colorable. Now, to complete the proof, it merely needs to show that  $d(s_r h')$  is  $G$ -colorable. Our argument proceeds as follows.

Write  $F = s_r h'$  and  $s_r = \rho_{n-k}^* + \delta_{n-k}$  where  $\delta_{n-k} \in V_h$ . As is known as above, after reducing modulo  $V_h$ ,  $h'$  contains no degree-one factor  $\rho_{n-k}^*$ . Thus, after reducing modulo  $V_h$ ,  $F$  becomes a squarefree homogeneous polynomial of degree  $k + 2$  in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}/V_h]$ . Similarly to the argument of Lemma 5.4, up to those automorphisms  $\sigma \in \text{Aut}(\widehat{\text{Hom}(\mathbb{Z}_2, G)})$  such that the restriction of  $\sigma$  to  $V_h$  is the identity, each monomial of  $F$  is of the following form

$$(\rho_{n-k}^* + \delta_{n-k})(\rho_{n-k+1}^* + \delta_{n-k+1}) \cdots (\rho_n^* + \delta_n)(\rho_{n-k}^* + \rho_{n-k+1}^* + \dots + \rho_{n-k+j}^* + \delta)$$

where  $1 \leq j \leq k$ , and  $\delta, \delta_i \in V_h$ ,  $i = n - k, n - k + 1, \dots, n$ . Furthermore, we can conclude that  $F$  is a linear combination of the polynomials  $\sigma(F_j)$ , where

$F_j = (\rho_{n-k}^* + \delta_{n-k})(\rho_{n-k+1}^* + \delta_{n-k+1}) \cdots (\rho_{n-k+j}^* + \delta_{n-k+j})(\rho_{n-k}^* + \rho_{n-k+1}^* + \cdots + \rho_{n-k+j}^* + \delta)b_j$  with  $b_j \in \mathcal{S}$  if  $j < k$  and  $b_k = 1$  if  $j = k$ , and  $\sigma$  is chosen in  $\text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$  such that the restriction of  $\sigma$  to  $V_h$  is the identity. In a similar way to Remark 10, we see easily that  $d((\rho_{n-k}^* + \delta_{n-k})(\rho_{n-k+1}^* + \delta_{n-k+1}) \cdots (\rho_{n-k+j}^* + \delta_{n-k+j})(\rho_{n-k}^* + \rho_{n-k+1}^* + \cdots + \rho_{n-k+j}^* + \delta))$  is  $G$ -colorable. When  $j < k$ , we have that  $\deg b_j = k - j < \deg h = k + 1$ , so by induction hypothesis again,  $b_j$  is  $G$ -colorable. Then by Lemma 3.8 we obtain that for  $1 \leq j \leq k$ ,  $d(F_j) = d((\rho_{n-k}^* + \delta_{n-k})(\rho_{n-k+1}^* + \delta_{n-k+1}) \cdots (\rho_{n-k+j}^* + \delta_{n-k+j})(\rho_{n-k}^* + \rho_{n-k+1}^* + \cdots + \rho_{n-k+j}^* + \delta))b_j$  is  $G$ -colorable. Moreover, since  $d(F) = d(s_r h')$  is a linear combination of the polynomials  $d(\sigma(F_j)) = \sigma(d(F_j))$ , by Remark 11 we conclude that  $d(s_r h')$  is  $G$ -colorable. This completes the proof.  $\square$

**Proposition 5.7.** *Each polynomial of  $\mathcal{V}_n^*$  is  $G$ -colorable.*

*Proof.* Let  $h$  be a polynomial of  $\mathcal{V}_n^*$ . By Lemma 5.2, there is a polynomial  $H$  in  $\mathcal{L}_n$  such that  $d(H) = h$ . Then by Lemma 5.4,  $d(H)$  is a linear combination of the polynomials  $d(\sigma(H_i))$ ,  $\sigma \in \text{Aut}(\text{Hom}(\mathbb{Z}_2, G))$ , where each  $H_i = \rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*)h_i$  as stated in Lemma 5.4. Since  $d(\sigma(H_i)) = \sigma(d(H_i))$ , by Remark 11 it suffices to show that each  $d(H_i)$  is  $G$ -colorable. We have known from Remark 10 that  $d(\rho_1^* \cdots \rho_i^*(\rho_1^* + \cdots + \rho_i^*))$  is the coloring polynomial of the colored  $i$ -simplex  $(\Delta^i, \lambda_0)$  with  $i + 1$  facets colored by  $\rho_1^*, \dots, \rho_i^*, \rho_1^* + \cdots + \rho_i^*$  respectively, so it is  $G$ -colorable. For  $i = n$ , we have that  $h_n = 1$  so  $d(H_n)$  is  $G$ -colorable. For  $i < n$ , we have that  $h_i \in \mathcal{S}$ , so  $h_i$  is  $G$ -colorable by Lemma 5.6. Thus, by Lemma 3.8 we can attain that  $d(H_i)$  is also  $G$ -colorable.  $\square$

**Corollary 5.8.**  $\mathfrak{M}_n$  is generated by the classes of small covers over  $\Delta^{n_1} \times \cdots \times \Delta^{n_\ell}$  where  $n_1 + \cdots + n_\ell = n$ .

*Proof.* We note by [8, §1.5. The basic construction] that each  $G$ -colored simple convex  $n$ -polytope  $(P^n, \lambda)$  can reconstruct an  $n$ -dimensional small cover  $M^n$  over  $P^n$ . So we have by Propositions 3.3 and 3.6 that  $\phi_n(\{M^n\})$  is the dual polynomial of  $g_{(P^n, \lambda)}$ . Then Corollary 5.8 immediately follows from Propositions 5.1 and 5.7.  $\square$

Now we can first give the proof of Theorem 1.5.

*Proof of Theorem 1.5.* This is a direct consequence of Corollary 5.8 and Lemma 3.8.  $\square$

Next let us prove Theorem 1.4.

*Proof of Theorem 1.4.* We have known from the proof of Corollary 5.8 that the  $G$ -coloring polynomial of each  $G$ -colored simple convex  $n$ -polytope  $(P^n, \lambda)$  uniquely determines an equivariant unoriented cobordism class containing a small cover over  $P^n$  as its representative. Thus, by Proposition 5.1 and Remark 8 we only need to prove that each polynomial in  $\mathcal{V}_n^*$  is the  $G$ -coloring polynomial of a  $G$ -colored simple convex  $n$ -polytope. In order to prove this, by Proposition 5.7 it suffices to show that

**Claim A.** *Let  $f_1$  and  $f_2$  be two polynomials in  $\mathcal{V}_n^*$ . If  $f_1$  and  $f_2$  are the coloring polynomials of two  $G$ -colored simple convex  $n$ -polytopes  $(P_1, \lambda_1)$  and  $(P_2, \lambda_2)$  respectively, then  $f_1 + f_2$  is the coloring polynomial of a  $G$ -colored simple convex  $n$ -polytope, too.*

If  $f_1$  and  $f_2$  have the same monomial  $s_1 \cdots s_n$ , then there must be a vertex  $v_1$  of  $P_1$  and a vertex  $v_2$  of  $P_2$  such that the coloring monomials of  $v_1$  and  $v_2$  are same and equal to  $s_1 \cdots s_n$ , i.e.,  $\lambda_{1v_1} = \lambda_{2v_2} = s_1 \cdots s_n$ . Now by Lemma 3.10 we can perform a connected sum  $P_1 \#_{v_1, v_2} P_2$  of  $(P_1, \lambda_1)$  and  $(P_2, \lambda_2)$  at  $v_1$  and  $v_2$ , such that  $P_1 \#_{v_1, v_2} P_2$  is a simple convex polytope and naturally admits a  $G$ -coloring  $\lambda_1 \#_{v_1, v_2} \lambda_2$  induced by  $\lambda_1$  and  $\lambda_2$ , and in particular,  $f_1 + f_2$  is the coloring polynomial of  $(P_1 \#_{v_1, v_2} P_2, \lambda_1 \#_{v_1, v_2} \lambda_2)$ .

Suppose that  $f_1$  and  $f_2$  contain no same monomial. If  $f_1$  contains a monomial  $s_1 s_2 \cdots s_n$  but  $f_2$  contains a monomial  $\tilde{s}_1 s_2 \cdots s_n$ , then there must be a vertex  $v_1$  of  $P_1$  and a vertex  $v_2$  of  $P_2$  such that  $\lambda_{1v_1} = s_1 s_2 \cdots s_n$  and  $\lambda_{2v_2} = \tilde{s}_1 s_2 \cdots s_n$ . Consider the colored simple convex polytope  $Q_1 = \Delta^{n-1} \times \Delta^1$  with coloring  $\lambda'$  such that  $n$  facets  $F_1 \times \Delta^1, \dots, F_n \times \Delta^1$  are colored by  $s_2, \dots, s_n, s_2 + \cdots + s_n$  and two facets  $\Delta^{n-1} \times *_1, \Delta^{n-1} \times *_2$  are colored by  $s_1, \tilde{s}_1$ , where  $F_1, \dots, F_n$  denote all facets of  $\Delta^{n-1}$  and  $*_1, *_2$  denote two facets of  $\Delta^1$ . Obviously, there are two vertices  $u_1, u_2$  of  $Q_1$  such that  $\lambda'_{u_1} = s_1 s_2 \cdots s_n$  and  $\lambda'_{u_2} = \tilde{s}_1 s_2 \cdots s_n$ . Choose a vertex  $q_1$  of  $Q_1$  such that  $\lambda'_{q_1} = s_1 s_3 \cdots s_n (s_2 + \cdots + s_n)$ . Then we can perform a connected sum of  $P_1, P_2, Q_1$  to get the following simple convex polytope

$$P = P_1 \#_{v_1, u_1} Q_1 \#_{q_1, q_1} Q_1 \#_{u_2, v_2} P_2$$

which admits a  $G$ -coloring  $\lambda$  determined by  $\lambda_1, \lambda_2, \lambda'$ . Furthermore, by Lemma 3.10 we see that  $f_1 + f_2$  is the coloring polynomial of  $(P, \lambda)$ .

Generally, if  $f_1$  contains a monomial  $s_1 s_2 \cdots s_n$  but  $f_2$  contains a monomial  $\tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_n$ , using the above constructed colored polytope  $(Q_1, \lambda')$ , we first can obtain a colored simple convex polytope  $(P_1 \#_{v_1, u_1} Q_1 \#_{q_1, q_1} Q_1, \lambda^{(1)})$  such that there is a vertex  $p_1$  with its coloring monomial  $\lambda_{p_1}^{(1)} = \tilde{s}_1 s_2 \cdots s_n$ . In the same way as above, we construct a colored simple convex polytope  $Q_2 = \Delta^{n-1} \times \Delta^1$  with coloring  $\lambda''$  such that  $n$  facets  $F_1 \times \Delta^1, \dots, F_n \times \Delta^1$  are colored by  $\tilde{s}_1, s_3, \dots, s_n, \tilde{s}_1 + s_3 + \cdots + s_n$  and two facets  $\Delta^{n-1} \times *_1, \Delta^{n-1} \times *_2$  are colored by  $s_2, \tilde{s}_2$ . In particular, it is not difficult to see that there are two vertices  $p'_1$  and  $p''_1$  such that  $\lambda''_{p'_1} = \tilde{s}_1 s_2 \cdots s_n$  and  $\lambda''_{p''_1} = \tilde{s}_1 \tilde{s}_2 s_3 \cdots s_n$ . Take the vertex  $q_2$  in  $Q_2$  such that  $\lambda''_{q_2} = s_2 \cdots s_n (\tilde{s}_1 + s_3 + \cdots + s_n)$ . Now by doing connected sum we can construct the following colored simple convex polytope

$$(P_1 \#_{v_1, u_1} Q_1 \#_{q_1, q_1} Q_1 \#_{p_1, p'_1} Q_2 \#_{q_2, q_2} Q_2, \lambda^{(2)})$$

such that there is a vertex  $p_2$  with  $\lambda_{p_2}^{(2)} = \tilde{s}_1 \tilde{s}_2 s_3 \cdots s_n$ . Continuing this procedure, we can further construct a series of colored simple convex polytopes  $Q_3, \dots, Q_n$ , so that finally we can obtain a colored simple convex polytope

$$P' = P_1 \#_{v_1, u_1} Q_1 \#_{q_1, q_1} Q_1 \#_{p_1, p'_1} Q_2 \#_{q_2, q_2} Q_2 \#_{p_2, p'_2} \cdots \#_{p_{n-1}, p'_{n-1}} Q_n \#_{q_n, q_n} Q_n$$

with  $G$ -coloring  $\lambda^{(n)}$  such that there is a vertex  $p_n$  of  $P'$  with  $\lambda_{p_n}^{(n)} = \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_n$ . Now let  $v_2$  be the vertex of  $P_2$  such that  $\lambda_{2v_2} = \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_n$ . Then we can get a colored simple convex polytope  $(P' \#_{p_n, v_2} P_2, \lambda)$  as desired. Thus, Claim A holds. The proof of Theorem 1.4 is completed.  $\square$

**Corollary 5.9.** *A faithful  $G$ -polynomial  $g \in \mathbb{Z}[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$  belongs to  $\text{Im } \phi_n$  if and only if its dual polynomial  $g^*$  is the  $G$ -coloring polynomial of a  $G$ -colored simple convex polytope  $(P^n, \lambda)$ .*

**5.3. Determination of  $\mathfrak{M}_4$ .** By Proposition 5.1 and Remark 8, it suffices to determine the structure of  $\mathcal{V}_4^*$ .

**Lemma 5.10.**  $\mathcal{V}_4^*$  is generated by the polynomials of the form

$$d(s_1 s_2 (s_1 + s_2)) d(s_3 s_4 (s_3 + s_4 + \varepsilon s_1))$$

where  $\{s_1, s_2, s_3, s_4\}$  is a basis of  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4)$  and  $\varepsilon = 0$  or  $1$ .

*Proof.* By Proposition 5.7,  $\mathcal{V}_4^*$  is generated by coloring polynomials of colored polytopes  $\Delta^4, \Delta^3 \times \Delta^1$  and  $\Delta^2 \times \Delta^2$ . By direct calculations, we may obtain that the coloring polynomials of colored polytopes  $\Delta^4, \Delta^3 \times \Delta^1$  and  $\Delta^2 \times \Delta^2$  are of the forms

$$\begin{cases} h_1 = d(s_1 s_2 s_3 s_4 (s_1 + s_2 + s_3 + s_4)) \\ h_2 = d(s_1 s_2 s_3 s_4 (s_1 + s_2 + s_3) + s_1 s_2 s_3 (s_1 + s_2 + s_3) (s_4 + s_1 + a s_2)) \\ h_3 = d(s_1 s_2 (s_1 + s_2)) d(s_3 s_4 (s_3 + s_4 + \varepsilon s_1)) \end{cases}$$

respectively, where  $a, \varepsilon = 0$  or  $1$ . Set  $\Lambda_i = \{\sigma(h_i) | \sigma \in \text{Aut}(\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4))\}$ . Now, to complete the proof, it suffices to prove that

**Claim B.** For each  $h \in \Lambda_1 \cup \Lambda_2$ ,  $h$  is a linear combination of polynomials in  $\Lambda_3$ .

Up to automorphisms of  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4)$ , this is equivalent to showing that  $h_1$  and  $h_2$  can be expressed as linear combinations of polynomials in  $\Lambda_3$ . By direct calculations we have that  $h_1 = h_{11} + h_{12}$  where

$$\begin{cases} h_{11} = d(s_1 (s_1 + s_2) s_3 s_4 (s_1 + s_2 + s_3 + s_4) + s_2 (s_1 + s_2) s_3 s_4 (s_1 + s_2 + s_3 + s_4)) \\ h_{12} = d(s_1 s_2 (s_1 + s_2)) d(s_3 s_4 (s_1 + s_2 + s_3 + s_4)). \end{cases}$$

and

$$h_2 = h_{21} + h_{22} + \begin{cases} h_{23} + h_{24} & \text{if } a = 1 \\ h'_{23} + h'_{24} + h_{25} + h_{26} & \text{if } a = 0 \end{cases}$$

where

$$\begin{cases} h_{21} = d(s_2 s_3 (s_2 + s_3)) d(s_1 s_4 (s_1 + s_4 + a s_2)) \\ h_{22} = d(s_2 s_3 (s_2 + s_3)) d(s_4 (s_1 + s_2 + s_3) (s_4 + (s_1 + s_2 + s_3) + a s_2 + s_2 + s_3)) \\ h_{23} = d(s_1 (s_2 + s_3) (s_1 + s_2 + s_3)) d(s_3 s_4 (s_3 + s_4 + (s_1 + s_2 + s_3))) \\ h_{24} = d(s_1 (s_2 + s_3) (s_1 + s_2 + s_3)) d(s_2 s_4 (s_2 + s_4 + s_1)) \\ h'_{23} = d(s_1 (s_2 + s_3) (s_1 + s_2 + s_3)) d(s_3 s_4 (s_3 + s_4)) \\ h'_{24} = d(s_1 (s_2 + s_3) (s_1 + s_2 + s_3)) d(s_2 s_4 (s_2 + s_4 + (s_2 + s_3))) \\ h_{25} = d(s_1 (s_2 + s_3) (s_1 + s_2 + s_3)) d(s_2 (s_1 + s_4) (s_2 + (s_1 + s_4) + (s_1 + s_2 + s_3))) \\ h_{26} = d(s_1 (s_2 + s_3) (s_1 + s_2 + s_3)) d(s_3 (s_1 + s_4) (s_3 + (s_1 + s_4) + s_1)). \end{cases}$$

It is easy to check that  $h_{11} \in \Lambda_2$  and  $h_{12}, h_{21}, h_{22}, h_{23}, h_{24}, h'_{23}, h'_{24}, h_{25}, h_{26} \in \Lambda_3$ . This proves Claim B, and thus we complete the proof of Lemma 5.10.  $\square$

*Proof of Proposition 1.6.* Each polynomial of the form

$$d(s_1 s_2 (s_1 + s_2)) d(s_3 s_4 (s_3 + s_4 + \varepsilon s_1))$$

is the coloring polynomial of a colored polytope  $\Delta^2 \times \Delta^2$ , where  $\{s_1, s_2, s_3, s_4\}$  is a basis of  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4)$  and  $\varepsilon = 0$  or  $1$ . Then it immediately follows from Lemma 5.10 that  $\mathfrak{M}_4$  is generated by the classes of small covers over  $\Delta^2 \times \Delta^2$ .



Now let us consider the dimension of  $\mathfrak{M}_4$ . By  $\mathcal{W}_4$  we denote the linear space generated by those degree-four monomials of  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4)}]$  whose dual monomials are all faithful. Then  $\dim_{\mathbb{Z}_2} \mathcal{W}_4$  is equal to the number of all bases in  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4) \cong (\mathbb{Z}_2)^4$ , so by [15, Remark 2.1],  $\dim_{\mathbb{Z}_2} \mathcal{W}_4 = 840$ . As a subspace of  $\mathcal{W}_4$ , by Lemma 5.10  $\mathcal{V}_4^*$  is generated by all polynomials of the form  $d(s_1 s_2 (s_1 + s_2)) d(s_3 s_4 (s_3 + s_4 + \varepsilon s_1))$ . Thus, we need to determine a maximal linearly independent subset of all polynomials of the form  $d(s_1 s_2 (s_1 + s_2)) d(s_3 s_4 (s_3 + s_4 + \varepsilon s_1))$  in  $\mathcal{W}_4$ . To do this, we can write a computer program to find such a maximal linearly independent subset, which contains 510 polynomials and is listed as follows:

1	(1; 14; 15; 6; 11; 13)	(1; 14; 15; 7; 10; 13)	(1; 14; 15; 4; 9; 13)	(1; 14; 15; 5; 8; 13)	(1; 14; 15; 7; 11; 12)
2	(1; 14; 15; 6; 10; 12)	(1; 14; 15; 5; 9; 12)	(1; 14; 15; 4; 8; 12)	(1; 14; 15; 2; 9; 11)	(1; 14; 15; 3; 8; 11)
3	(1; 14; 15; 3; 9; 10)	(1; 14; 15; 2; 8; 10)	(1; 14; 15; 2; 5; 7)	(1; 14; 15; 3; 4; 7)	(1; 14; 15; 3; 5; 6)
4	(1; 14; 15; 2; 4; 6)	(2; 13; 15; 7; 11; 12)	(2; 13; 15; 6; 10; 12)	(2; 13; 15; 5; 9; 12)	(2; 13; 15; 4; 8; 12)
5	(1; 10; 11; 2; 13; 15)	(2; 13; 15; 3; 8; 11)	(2; 13; 15; 5; 11; 14)	(2; 13; 15; 3; 9; 10)	(2; 13; 15; 4; 10; 14)
6	(1; 8; 9; 2; 13; 15)	(2; 13; 15; 7; 9; 14)	(2; 13; 15; 6; 8; 14)	(1; 6; 7; 2; 13; 15)	(2; 13; 15; 3; 4; 7)
7	(2; 13; 15; 3; 5; 6)	(1; 4; 5; 2; 13; 15)	(1; 10; 11; 3; 12; 15)	(2; 9; 11; 3; 12; 15)	(3; 12; 15; 6; 11; 13)
8	(3; 12; 15; 5; 11; 14)	(2; 8; 10; 3; 12; 15)	(3; 12; 15; 7; 10; 13)	(3; 12; 15; 4; 10; 14)	(1; 8; 9; 3; 12; 15)
9	(3; 12; 15; 7; 9; 14)	(3; 12; 15; 4; 9; 13)	(3; 12; 15; 6; 8; 14)	(3; 12; 15; 5; 8; 13)	(1; 6; 7; 3; 12; 15)
10	(2; 5; 7; 3; 12; 15)	(2; 4; 6; 3; 12; 15)	(1; 4; 5; 3; 12; 15)	(3; 9; 10; 4; 11; 15)	(2; 8; 10; 4; 11; 15)
11	(4; 11; 15; 7; 10; 13)	(4; 11; 15; 6; 10; 12)	(1; 8; 9; 4; 11; 15)	(4; 11; 15; 7; 9; 14)	(4; 11; 15; 5; 9; 12)
12	(4; 11; 15; 6; 8; 14)	(4; 11; 15; 5; 8; 13)	(1; 6; 7; 4; 11; 15)	(2; 5; 7; 4; 11; 15)	(3; 5; 6; 4; 11; 15)
13	(1; 2; 3; 4; 11; 15)	(1; 8; 9; 5; 10; 15)	(5; 10; 15; 7; 9; 14)	(4; 9; 13; 5; 10; 15)	(2; 9; 11; 5; 10; 15)
14	(5; 10; 15; 6; 8; 14)	(4; 8; 12; 5; 10; 15)	(3; 8; 11; 5; 10; 15)	(1; 6; 7; 5; 10; 15)	(3; 4; 7; 5; 10; 15)
15	(2; 4; 6; 5; 10; 15)	(1; 2; 3; 5; 10; 15)	(5; 8; 13; 6; 9; 15)	(4; 8; 12; 6; 9; 15)	(3; 8; 11; 6; 9; 15)
16	(2; 8; 10; 6; 9; 15)	(2; 5; 7; 6; 9; 15)	(3; 4; 7; 6; 9; 15)	(1; 4; 5; 6; 9; 15)	(1; 2; 3; 6; 9; 15)
17	(3; 5; 6; 7; 8; 15)	(2; 4; 6; 7; 8; 15)	(1; 4; 5; 7; 8; 15)	(1; 2; 3; 7; 8; 15)	(3; 13; 14; 7; 11; 12)
18	(3; 13; 14; 6; 10; 12)	(3; 13; 14; 5; 9; 12)	(3; 13; 14; 4; 8; 12)	(1; 10; 11; 3; 13; 14)	(2; 9; 11; 3; 13; 14)
19	(3; 13; 14; 4; 11; 15)	(2; 8; 10; 3; 13; 14)	(3; 13; 14; 5; 10; 15)	(1; 8; 9; 3; 13; 14)	(3; 13; 14; 6; 9; 15)
20	(3; 13; 14; 7; 8; 15)	(1; 6; 7; 3; 13; 14)	(2; 5; 7; 3; 13; 14)	(2; 4; 6; 3; 13; 14)	(1; 4; 5; 3; 13; 14)
21	(1; 10; 11; 2; 12; 14)	(2; 12; 14; 3; 8; 11)	(2; 12; 14; 6; 11; 13)	(2; 12; 14; 4; 11; 15)	(2; 12; 14; 3; 9; 10)
22	(2; 12; 14; 7; 10; 13)	(2; 12; 14; 5; 10; 15)	(1; 8; 9; 2; 12; 14)	(2; 12; 14; 6; 9; 15)	(2; 12; 14; 4; 9; 13)
23	(2; 12; 14; 7; 8; 15)	(2; 12; 14; 5; 8; 13)	(1; 6; 7; 2; 12; 14)	(2; 12; 14; 3; 4; 7)	(2; 12; 14; 3; 5; 6)
24	(1; 4; 5; 2; 12; 14)	(3; 9; 10; 5; 11; 14)	(2; 8; 10; 5; 11; 14)	(5; 11; 14; 7; 10; 13)	(5; 11; 14; 6; 10; 12)
25	(1; 8; 9; 5; 11; 14)	(5; 11; 14; 6; 9; 15)	(4; 9; 13; 5; 11; 14)	(5; 11; 14; 7; 8; 15)	(4; 8; 12; 5; 11; 14)
26	(1; 6; 7; 5; 11; 14)	(3; 4; 7; 5; 11; 14)	(2; 4; 6; 5; 11; 14)	(1; 2; 3; 5; 11; 14)	(1; 8; 9; 4; 10; 14)
27	(4; 10; 14; 6; 9; 15)	(4; 10; 14; 5; 9; 12)	(2; 9; 11; 4; 10; 14)	(4; 10; 14; 7; 8; 15)	(4; 10; 14; 5; 8; 13)
28	(3; 8; 11; 4; 10; 14)	(1; 6; 7; 4; 10; 14)	(2; 5; 7; 4; 10; 14)	(3; 5; 6; 4; 10; 14)	(1; 2; 3; 4; 10; 14)
29	(5; 8; 13; 7; 9; 14)	(4; 8; 12; 7; 9; 14)	(3; 8; 11; 7; 9; 14)	(2; 8; 10; 7; 9; 14)	(3; 5; 6; 7; 9; 14)
30	(2; 4; 6; 7; 9; 14)	(1; 4; 5; 7; 9; 14)	(1; 2; 3; 7; 9; 14)	(2; 5; 7; 6; 8; 14)	(3; 4; 7; 6; 8; 14)
31	(1; 4; 5; 6; 8; 14)	(1; 2; 3; 6; 8; 14)	(1; 12; 13; 2; 9; 11)	(1; 12; 13; 3; 8; 11)	(1; 12; 13; 5; 11; 14)
32	(1; 12; 13; 4; 11; 15)	(1; 12; 13; 3; 9; 10)	(1; 12; 13; 2; 8; 10)	(1; 12; 13; 5; 10; 15)	(1; 12; 13; 4; 10; 14)
33	(1; 12; 13; 7; 9; 14)	(1; 12; 13; 6; 9; 15)	(1; 12; 13; 7; 8; 15)	(1; 12; 13; 6; 8; 14)	(1; 12; 13; 2; 5; 7)
34	(1; 12; 13; 3; 4; 7)	(1; 12; 13; 3; 5; 6)	(1; 12; 13; 2; 4; 6)	(3; 9; 10; 6; 11; 13)	(2; 8; 10; 6; 11; 13)
35	(5; 10; 15; 6; 11; 13)	(4; 10; 14; 6; 11; 13)	(1; 8; 9; 6; 11; 13)	(6; 11; 13; 7; 9; 14)	(5; 9; 12; 6; 11; 13)
36	(6; 11; 13; 7; 8; 15)	(4; 8; 12; 6; 11; 13)	(2; 5; 7; 6; 11; 13)	(3; 4; 7; 6; 11; 13)	(1; 4; 5; 6; 11; 13)
37	(1; 2; 3; 6; 11; 13)	(1; 8; 9; 7; 10; 13)	(6; 9; 15; 7; 10; 13)	(5; 9; 12; 7; 10; 13)	(2; 9; 11; 7; 10; 13)
38	(6; 8; 14; 7; 10; 13)	(4; 8; 12; 7; 10; 13)	(3; 8; 11; 7; 10; 13)	(3; 5; 6; 7; 10; 13)	(2; 4; 6; 7; 10; 13)
39	(1; 4; 5; 7; 10; 13)	(1; 2; 3; 7; 10; 13)	(4; 9; 13; 7; 8; 15)	(4; 9; 13; 6; 8; 14)	(3; 8; 11; 4; 9; 13)
40	(2; 8; 10; 4; 9; 13)	(1; 6; 7; 4; 9; 13)	(2; 5; 7; 4; 9; 13)	(3; 5; 6; 4; 9; 13)	(1; 2; 3; 4; 9; 13)
41	(1; 6; 7; 5; 8; 13)	(3; 4; 7; 5; 8; 13)	(2; 4; 6; 5; 8; 13)	(1; 2; 3; 5; 8; 13)	(3; 9; 10; 7; 11; 12)
42	(2; 8; 10; 7; 11; 12)	(5; 10; 15; 7; 11; 12)	(4; 10; 14; 7; 11; 12)	(1; 8; 9; 7; 11; 12)	(6; 9; 15; 7; 11; 12)
43	(4; 9; 13; 7; 11; 12)	(6; 8; 14; 7; 11; 12)	(5; 8; 13; 7; 11; 12)	(3; 5; 6; 7; 11; 12)	(2; 4; 6; 7; 11; 12)
44	(1; 4; 5; 7; 11; 12)	(1; 2; 3; 7; 11; 12)	(1; 8; 9; 6; 10; 12)	(6; 10; 12; 7; 9; 14)	(4; 9; 13; 6; 10; 12)
45	(2; 9; 11; 6; 10; 12)	(6; 10; 12; 7; 8; 15)	(5; 8; 13; 6; 10; 12)	(3; 8; 11; 6; 10; 12)	(2; 5; 7; 6; 10; 12)
46	(1; 4; 5; 6; 10; 12)	(1; 2; 3; 6; 10; 12)	(5; 9; 12; 7; 8; 15)	(3; 8; 11; 5; 9; 12)	(2; 8; 10; 5; 9; 12)
47	(1; 6; 7; 5; 9; 12)	(1; 2; 3; 5; 9; 12)	(1; 6; 7; 4; 8; 12)	(1; 2; 3; 4; 8; 12)	(1; 10; 11; 7; 9; 14)
48	(1; 10; 11; 6; 9; 15)	(1; 10; 11; 5; 9; 12)	(1; 10; 11; 4; 9; 13)	(1; 10; 11; 7; 8; 15)	(1; 10; 11; 6; 8; 14)
49	(1; 10; 11; 5; 8; 13)	(1; 10; 11; 4; 8; 12)	(1; 10; 11; 2; 5; 7)	(1; 10; 11; 3; 4; 7)	(1; 10; 11; 3; 5; 6)
50	(1; 10; 11; 2; 4; 6)	(2; 9; 11; 7; 8; 15)	(2; 9; 11; 6; 8; 14)	(1; 6; 7; 2; 9; 11)	(2; 9; 11; 3; 4; 7)
51	(2; 9; 11; 3; 5; 6)	(1; 4; 5; 2; 9; 11)	(1; 6; 7; 3; 8; 11)	(2; 5; 7; 3; 8; 11)	(2; 4; 6; 3; 8; 11)
52	(1; 4; 5; 3; 8; 11)	(3; 9; 10; 7; 8; 15)	(1; 6; 7; 3; 9; 10)	(1; 6; 7; 2; 8; 10)	(1; 8; 9; 2; 5; 7)
53	(1; 14; 15; 9; 11; 13)	(1; 14; 15; 8; 10; 13)	(1; 14; 15; 5; 7; 13)	(1; 14; 15; 4; 6; 13)	(1; 14; 15; 8; 11; 12)
54	(1; 14; 15; 9; 10; 12)	(1; 14; 15; 4; 7; 12)	(1; 14; 15; 5; 6; 12)	(1; 14; 15; 3; 7; 11)	(1; 14; 15; 2; 6; 11)
55	(1; 14; 15; 2; 7; 10)	(1; 14; 15; 3; 6; 10)	(2; 13; 15; 8; 11; 12)	(2; 13; 15; 9; 10; 12)	(2; 13; 15; 4; 7; 12)
56	(2; 13; 15; 5; 6; 12)	(2; 13; 15; 10; 11; 14)	(2; 13; 15; 3; 7; 11)	(2; 13; 15; 1; 5; 11)	(2; 13; 15; 3; 6; 10)
57	(2; 13; 15; 1; 4; 10)	(2; 13; 15; 8; 9; 14)	(2; 13; 15; 1; 7; 9)	(2; 13; 15; 1; 6; 8)	(3; 12; 15; 10; 11; 14)
58	(3; 12; 15; 9; 11; 13)	(3; 12; 15; 2; 6; 11)	(3; 12; 15; 1; 5; 11)	(3; 12; 15; 8; 10; 13)	(3; 12; 15; 2; 7; 10)
59	(3; 12; 15; 1; 4; 10)	(3; 12; 15; 8; 9; 14)	(3; 12; 15; 1; 7; 9)	(3; 12; 15; 2; 4; 9)	(3; 12; 15; 1; 6; 8)
60	(3; 12; 15; 2; 5; 8)	(4; 11; 15; 9; 10; 12)	(4; 11; 15; 8; 10; 13)	(4; 11; 15; 2; 7; 10)	(4; 11; 15; 3; 6; 10)
61	(4; 11; 15; 8; 9; 14)	(4; 11; 15; 1; 7; 9)	(4; 11; 15; 3; 5; 9)	(4; 11; 15; 1; 6; 8)	(4; 11; 15; 2; 5; 8)
62	(4; 11; 15; 2; 3; 14)	(4; 11; 15; 1; 3; 13)	(4; 11; 15; 1; 2; 12)	(5; 10; 15; 8; 9; 14)	(5; 10; 15; 1; 7; 9)

63	(5; 10; 15; 2; 4; 9)	(5; 10; 15; 1; 6; 8)	(5; 10; 15; 3; 4; 8)	(5; 10; 15; 4; 7; 12)	(5; 10; 15; 3; 7; 11)
64	(5; 10; 15; 4; 6; 13)	(5; 10; 15; 2; 6; 11)	(5; 10; 15; 2; 3; 14)	(5; 10; 15; 1; 3; 13)	(5; 10; 15; 1; 2; 12)
65	(6; 9; 15; 2; 5; 8)	(6; 9; 15; 3; 4; 8)	(6; 9; 15; 5; 7; 13)	(6; 9; 15; 4; 7; 12)	(6; 9; 15; 3; 7; 11)
66	(6; 9; 15; 2; 7; 10)	(6; 9; 15; 4; 5; 14)	(6; 9; 15; 1; 5; 11)	(6; 9; 15; 1; 4; 10)	(6; 9; 15; 2; 3; 14)
67	(6; 9; 15; 1; 3; 13)	(6; 9; 15; 1; 2; 12)	(7; 8; 15; 5; 6; 12)	(7; 8; 15; 4; 6; 13)	(7; 8; 15; 3; 6; 10)
68	(7; 8; 15; 2; 6; 11)	(7; 8; 15; 4; 5; 14)	(7; 8; 15; 3; 5; 9)	(7; 8; 15; 1; 5; 11)	(7; 8; 15; 2; 4; 9)
69	(7; 8; 15; 1; 4; 10)	(7; 8; 15; 2; 3; 14)	(7; 8; 15; 1; 3; 13)	(7; 8; 15; 1; 2; 12)	(3; 13; 14; 9; 11; 12)
70	(3; 13; 14; 8; 10; 12)	(3; 13; 14; 5; 7; 12)	(3; 13; 14; 4; 6; 12)	(3; 13; 14; 10; 11; 15)	(3; 13; 14; 2; 7; 11)
71	(3; 13; 14; 1; 4; 11)	(3; 13; 14; 2; 6; 10)	(3; 13; 14; 1; 5; 10)	(3; 13; 14; 8; 9; 15)	(3; 13; 14; 1; 6; 9)
72	(3; 13; 14; 1; 7; 8)	(2; 12; 14; 10; 11; 15)	(2; 12; 14; 8; 11; 13)	(2; 12; 14; 3; 6; 11)	(2; 12; 14; 1; 4; 11)
73	(2; 12; 14; 1; 5; 10)	(2; 12; 14; 8; 9; 15)	(2; 12; 14; 1; 6; 9)	(2; 12; 14; 1; 7; 8)	(5; 11; 14; 9; 10; 13)
74	(5; 11; 14; 8; 10; 12)	(5; 11; 14; 3; 7; 10)	(5; 11; 14; 2; 6; 10)	(5; 11; 14; 8; 9; 15)	(5; 11; 14; 1; 6; 9)
75	(5; 11; 14; 3; 4; 9)	(5; 11; 14; 1; 7; 8)	(5; 11; 14; 2; 4; 8)	(5; 11; 14; 2; 3; 15)	(5; 11; 14; 1; 3; 12)
76	(5; 11; 14; 1; 2; 13)	(4; 10; 14; 8; 9; 15)	(4; 10; 14; 1; 6; 9)	(4; 10; 14; 2; 5; 9)	(4; 10; 14; 1; 7; 8)
77	(4; 10; 14; 2; 7; 11)	(4; 10; 14; 2; 3; 15)	(4; 10; 14; 1; 3; 12)	(4; 10; 14; 1; 2; 13)	(7; 9; 14; 3; 5; 8)
78	(7; 9; 14; 2; 4; 8)	(7; 9; 14; 5; 6; 13)	(7; 9; 14; 4; 6; 12)	(7; 9; 14; 3; 6; 11)	(7; 9; 14; 2; 6; 10)
79	(7; 9; 14; 4; 5; 15)	(7; 9; 14; 1; 5; 10)	(7; 9; 14; 1; 4; 11)	(7; 9; 14; 2; 3; 15)	(7; 9; 14; 1; 3; 12)
80	(7; 9; 14; 1; 2; 13)	(6; 8; 14; 5; 7; 12)	(6; 8; 14; 4; 5; 15)	(6; 8; 14; 2; 5; 9)	(6; 8; 14; 1; 5; 10)
81	(6; 8; 14; 1; 4; 11)	(6; 8; 14; 2; 3; 15)	(6; 8; 14; 1; 3; 12)	(6; 8; 14; 1; 2; 13)	(1; 12; 13; 9; 11; 15)
82	(1; 12; 13; 8; 11; 14)	(1; 12; 13; 3; 5; 11)	(1; 12; 13; 2; 4; 11)	(1; 12; 13; 9; 10; 14)	(1; 12; 13; 8; 10; 15)
83	(1; 12; 13; 2; 5; 10)	(1; 12; 13; 3; 4; 10)	(1; 12; 13; 3; 7; 9)	(1; 12; 13; 2; 6; 9)	(1; 12; 13; 2; 7; 8)
84	(1; 12; 13; 3; 6; 8)	(6; 11; 13; 9; 10; 14)	(6; 11; 13; 8; 10; 15)	(6; 11; 13; 2; 5; 10)	(6; 11; 13; 3; 4; 10)
85	(6; 11; 13; 8; 9; 12)	(6; 11; 13; 3; 7; 9)	(6; 11; 13; 1; 5; 9)	(6; 11; 13; 2; 7; 8)	(6; 11; 13; 1; 4; 8)
86	(6; 11; 13; 2; 3; 12)	(6; 11; 13; 1; 3; 15)	(6; 11; 13; 1; 2; 14)	(7; 10; 13; 8; 9; 12)	(7; 10; 13; 2; 6; 9)
87	(7; 10; 13; 1; 5; 9)	(7; 10; 13; 3; 6; 8)	(7; 10; 13; 1; 4; 8)	(7; 10; 13; 5; 6; 14)	(7; 10; 13; 4; 6; 15)
88	(7; 10; 13; 3; 5; 11)	(7; 10; 13; 2; 4; 11)	(7; 10; 13; 2; 3; 12)	(7; 10; 13; 1; 3; 15)	(7; 10; 13; 1; 2; 14)
89	(4; 9; 13; 2; 7; 8)	(4; 9; 13; 2; 3; 12)	(4; 9; 13; 1; 3; 15)	(4; 9; 13; 1; 2; 14)	(5; 8; 13; 6; 7; 12)
90	(5; 8; 13; 1; 3; 15)	(5; 8; 13; 1; 2; 14)	(7; 11; 12; 9; 10; 15)	(7; 11; 12; 8; 10; 14)	(7; 11; 12; 3; 5; 10)
91	(7; 11; 12; 2; 4; 10)	(7; 11; 12; 8; 9; 13)	(7; 11; 12; 3; 6; 9)	(7; 11; 12; 1; 4; 9)	(7; 11; 12; 2; 6; 8)
92	(7; 11; 12; 1; 5; 8)	(7; 11; 12; 2; 3; 13)	(7; 11; 12; 1; 3; 14)	(7; 11; 12; 1; 2; 15)	(6; 10; 12; 8; 9; 13)
93	(6; 10; 12; 2; 7; 9)	(6; 10; 12; 1; 4; 9)	(6; 10; 12; 2; 5; 11)	(6; 10; 12; 2; 3; 13)	(6; 10; 12; 1; 3; 14)
94	(6; 10; 12; 1; 2; 15)	(5; 9; 12; 2; 3; 13)	(5; 9; 12; 1; 3; 14)	(5; 9; 12; 1; 2; 15)	(1; 10; 11; 5; 7; 9)
95	(1; 10; 11; 4; 6; 9)	(1; 10; 11; 4; 7; 8)	(1; 10; 11; 5; 6; 8)	(1; 10; 11; 3; 7; 15)	(1; 10; 11; 2; 7; 14)
96	(1; 10; 11; 3; 6; 14)	(1; 10; 11; 2; 6; 15)	(1; 10; 11; 3; 5; 13)	(1; 10; 11; 2; 5; 12)	(1; 10; 11; 3; 4; 12)
97	(1; 10; 11; 2; 4; 13)	(2; 9; 11; 6; 7; 10)	(2; 9; 11; 1; 7; 13)	(2; 9; 11; 3; 6; 14)	(2; 9; 11; 1; 6; 12)
98	(2; 9; 11; 4; 5; 10)	(2; 9; 11; 3; 5; 13)	(2; 9; 11; 1; 5; 15)	(2; 9; 11; 3; 4; 12)	(2; 9; 11; 1; 4; 14)
99	(3; 8; 11; 2; 6; 15)	(3; 8; 11; 1; 5; 15)	(3; 8; 11; 2; 4; 13)	(3; 8; 11; 1; 4; 14)	(7; 11; 12; 5; 6; 8)
100	(6; 11; 13; 4; 5; 10)	(5; 11; 14; 3; 4; 12)	(5; 11; 14; 1; 3; 9)	(4; 11; 15; 2; 3; 10)	(4; 11; 15; 1; 3; 9)
101	(4; 11; 15; 1; 2; 8)	(3; 9; 10; 5; 7; 8)	(3; 9; 10; 4; 6; 8)	(3; 9; 10; 2; 7; 15)	(3; 9; 10; 2; 6; 14)
102	(2; 8; 10; 4; 7; 9)	(2; 8; 10; 3; 7; 14)	(7; 9; 14; 4; 5; 8)	(7; 9; 14; 3; 5; 15)	(7; 9; 14; 1; 5; 13)

In the above table, each  $(a; b; c; d; e; l)$  means a polynomial  $d(x_a x_b x_c) d(x_d x_e x_l)$  in  $\mathcal{V}_4^*$ , where  $x_1, x_2, \dots, x_{15}$  denote the 15 nontrivial elements in  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4)$  with

$$\begin{cases} x_1 = \rho_1^* & x_2 = \rho_2^* \\ x_3 = \rho_1^* + \rho_2^* & x_4 = \rho_3^* \\ x_5 = \rho_1^* + \rho_3^* & x_6 = \rho_2^* + \rho_3^* \\ x_7 = \rho_1^* + \rho_2^* + \rho_3^* & x_8 = \rho_4^* \\ x_9 = \rho_1^* + \rho_4^* & x_{10} = \rho_2^* + \rho_4^* \\ x_{11} = \rho_1^* + \rho_2^* + \rho_4^* & x_{12} = \rho_3^* + \rho_4^* \\ x_{13} = \rho_1^* + \rho_3^* + \rho_4^* & x_{14} = \rho_2^* + \rho_3^* + \rho_4^* \\ x_{15} = \rho_1^* + \rho_2^* + \rho_3^* + \rho_4^* & \{\rho_1^*, \rho_2^*, \rho_3^*, \rho_4^*\} \text{ is the standard basis of } \text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^4). \end{cases}$$

Thus we conclude that  $\dim_{\mathbb{Z}_2} \mathcal{V}_4^* = \dim_{\mathbb{Z}_2} \mathfrak{M}_4 = 510$ .  $\square$

*Remark 12.* In a similar way to Lemma 5.10, we conclude easily that  $\mathcal{V}_3^*$  is generated by the polynomials of the form

$$d(s_1 s_2 (s_1 + s_2)) d(s_3 (s_1 + s_3))$$

where  $\{s_1, s_2, s_3\}$  is a basis of  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^3)$ . So  $\mathfrak{M}_3$  is generated by the classes of small covers over  $\Delta^1 \times \Delta^2$ . We can also list a basis of  $\mathcal{V}_3^*$  as follows:

$d(x_1x_6x_7)d(x_2x_5)$	$d(x_1x_6x_7)d(x_3x_4)$	$d(x_2x_5x_7)d(x_3x_4)$	$d(x_2x_5x_7)d(x_1x_6)$
$d(x_3x_4x_7)d(x_2x_5)$	$d(x_3x_4x_7)d(x_1x_6)$	$d(x_3x_5x_6)d(x_2x_4)$	$d(x_3x_5x_6)d(x_1x_7)$
$d(x_2x_4x_6)d(x_1x_7)$	$d(x_1x_4x_5)d(x_3x_6)$	$d(x_1x_4x_5)d(x_2x_7)$	$d(x_2x_5x_7)d(x_1x_4)$
$d(x_3x_4x_7)d(x_2x_6)$			

where  $x_1, x_2, \dots, x_7$  denote the 7 nontrivial elements in  $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^3)$  with

$$\begin{cases} x_1 = \rho_1^* & x_2 = \rho_2^* \\ x_3 = \rho_1^* + \rho_2^* & x_4 = \rho_3^* \\ x_5 = \rho_1^* + \rho_3^* & x_6 = \rho_2^* + \rho_3^* \\ x_7 = \rho_1^* + \rho_2^* + \rho_3^* & \{\rho_1^*, \rho_2^*, \rho_3^*\} \text{ is the standard basis of } \text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^3). \end{cases}$$

Therefore, we obtain that  $\dim_{\mathbb{Z}_2} \mathfrak{M}_3 = 13$ , and  $\mathfrak{M}_3$  is generated by the classes of the  $S^1 \times \mathbb{R}P^2$ 's with effective  $(\mathbb{Z}_2)^3$ -actions. Note that as shown in [15], a nonbounding effective  $(\mathbb{Z}_2)^3$ -action on  $S^1 \times \mathbb{R}P^2$  with  $\Delta^1 \times \Delta^2$  as its orbit space can be constructed, and applying automorphisms of  $(\mathbb{Z}_2)^3$  to this effective action can give all required basis elements of  $\mathfrak{M}_3$ .

## 6. A SUMMARY AND FURTHER PROBLEMS

Together with Theorems 1.2, 3.1 and Corollaries 1.3, 5.9, we see that there are some essential relationships among 2-torus manifolds, coloring polynomials, colored simple convex polytopes, colored graphs, which are stated as follows:

**Theorem 6.1.** *Let  $g = \sum_i t_{i,1} \cdots t_{i,n}$  be a faithful  $G$ -polynomial in  $\mathbb{Z}_2[\widehat{\text{Hom}(G, \mathbb{Z}_2)}]$ . Then the following statements are all equivalent.*

- (1)  $g \in \text{Im } \phi_n$  (i.e., there is an  $n$ -dimensional 2-torus manifold  $M^n$  such that  $\sum_{p \in M^G} [\tau_p M] = g$ );
- (2)  $g$  is the  $G$ -coloring polynomial of a  $G$ -colored graph  $(\Gamma, \alpha)$ ;
- (3)  $g = \sum_i t_{i,1} \cdots t_{i,n}$  possesses the property that for any symmetric polynomial function  $f(x_1, \dots, x_n)$  over  $\mathbb{Z}_2$ ,

$$\sum_i \frac{f(t_{i,1}, \dots, t_{i,n})}{t_{i,1} \cdots t_{i,n}} \in \mathbb{Z}_2[\text{Hom}(G, \mathbb{Z}_2)];$$

- (4)  $d(g^*) = 0$ ;
- (5)  $g^*$  is the  $G$ -coloring polynomial of a  $G$ -colored simple convex polytope  $(P^n, \lambda)$

where  $g^*$  is the dual polynomial of  $g$ .

Based upon the above equivalent results, it seems to be interesting to discuss the properties of regular graphs and simple convex polytopes. We see by Theorem 1.4 and Remark 7 that  $\Gamma$  in Theorem 6.1(2) can actually be chosen as the 1-skeleton of a polytope. However, for a  $G$ -colored graph  $(\Gamma, \alpha)$ , we don't know when  $\Gamma$  will become the 1-skeleton of a polytope. Indeed, given a graph, to determine whether it is the 1-skeleton of a polytope or not is a quite difficult problem except for the known Steinitz theorem (see [11]). In addition, Corollary 3.9 gives a sufficient condition that a simple convex polytope with a coloring is indecomposable. These observations lead us to pose the following problems:

- (P1) For a  $G$ -colored graph  $(\Gamma, \alpha)$ , under what condition will  $\Gamma$  be the 1-skeleton of a polytope?

- (P2) Given a  $G$ -colored simple convex polytope  $(P^n, \lambda)$ , can we give a necessary and sufficient condition that  $P^n$  is indecomposable?

*Remark 13.* On the problem (P2), it is not difficult to see from the proof of Theorem 1.4 that if  $P^n$  is indecomposable, then  $g_{(P^n, \lambda)}$  may not be indecomposable in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, G)}]$ . However, if we add a restriction condition that the number of all monomials of  $g_{(P^n, \lambda)}$  is equal to that of all vertices of  $P^n$ , then an easy argument shows that when  $n = 3$ ,  $P^3$  is indecomposable if and only if  $g_{(P^3, \lambda)}$  is indecomposable in  $\mathbb{Z}_2[\widehat{\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^3)}]$ . It should be reasonable to conjecture that this is also true in the higher-dimensional case.

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